Goal: To understand the structure of neutron stars

[Problem 1] The ideal Fermi gas equation of states of white dwarfs or neutron stars, in which quantum degeneracy pressure dominates, can be represented by polytropic form;

$$P_{\rm deg} = K_\Gamma \rho^\Gamma = K_n \rho^{(n+1)/n}$$

where $K_\Gamma(K_n)$ and Γ are constants, ρ is density and n is called the polytropic index.

a) For the comparison, consider an adiabatic expansion of an ideal monoatomic gas for which thermal (kinetic) pressure, $P_{\rm kin} = (\rho/m)kT$, dominates and degeneracy pressure is negligible. Show that

$$TV^{\gamma-1} = \text{constant}, \quad P_{\rm kin}V^\gamma = \text{constant}, \quad P_{\rm kin} \propto \rho^\gamma$$

where $\gamma = c_P/c_V = 5/3$ is the ratio of specific heats (c_P: specific heat at constant pressure, c_V: specific heat at constant volume).

b) For ideal Fermi gas, in the zero temperature limit, show that the number density of gas (n_g) and Fermi momentum (p_F) are related by

$$n_g = g \times \frac{2\pi}{3h^3} p_F^3$$

where g is the degeneracy.

c) In the non-relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

d) In the full relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

e) For typical white dwarfs ($\rho \sim 10^6$ g cm^{-3}) and neutron stars ($\rho \sim 10^{14}$ g cm^{-3}), compare the magnitudes of degeneracy pressure ($P_{\rm deg}$), kinetic pressure of the ideal gas ($P_{\rm kin}$) and radiation pressure at $kT = 1$ MeV, 1 keV and 1 eV.

$$P_{\rm total} = P_{\rm deg} + \frac{\rho}{m}kT + \frac{1}{3}aT^4$$

where $k = 1.4 \times 10^{-16}$ erg K^{-1} $= 8.6 \times 10^{-5}$ eV K^{-1}.

[Problem 2] At zero temperature limit, when the compact star is in a hydrostatic equilibrium with spherical symmetry, compact star equation of state can be obtained by solving TOV (Tolman-Oppenheimer-Volkoff) equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2r^2}\left(1+\frac{P(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3P(r)}{M(r)c^2}\right)\left(1-\frac{2GM(r)}{c^2r}\right)^{-1}$$

where $P(r)$ is the pressure, $\epsilon(r)$ is the energy density and $M(r)$ is the enclosed gravitational mass $M_G(r)$ for a given radius r. The gravitational and baryon masses of the star are defined by

$$M_G(r) = \int_0^R 4\pi r^2\frac{\epsilon(r)}{c^2}$$

$$M_A(r) = m_A\int_0^R dr 4\pi r^2 n(r)\left(1-\frac{2GM(r)}{c^2r}\right)^{-1/2}$$

where m_A is baryon mass and $n(r)$ is the baryon number density.

a) In the Newtonian limit ($P \ll \epsilon$ and $GM/c^2 \ll r$), with polytropic EOS, show that TOV equation can be reduced to Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

with $\rho = \rho_c\theta^n$, $r = a\xi$, and

$$a = \sqrt{\frac{(n+1)K_n\rho_c^{(1-n)/n}}{4\pi G}},$$

where ρ_c is the central density of a star.

b) The Lane-Emden equation can be solved with boundary conditions at the center;

$$\theta(0) = 1, \quad \theta'(0) = 0.$$

For $n < 5$ (or $\Gamma > 6/5$), the solution decreases monotonically and have a zero at a finite value $\xi = \xi_1$: $\theta(\xi_1) = 0$ (see Table 1 for numerical values). This point corresponds to the surface of the star, where $P = \rho = 0$. Show that the mass of the star is given as

$$M = 4\pi\left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2}\rho_c^{(3-n)/2n}\left|\left(\xi^2\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right|.$$

c) In the full relativistic and Newtonian limit, show that the mass become independent of radius. This implies that there exist maximum mass (Chandrasekhar mass) for the compact stars (for which quantum degeneracy pressure dominates). What is the value of the radius-independent mass?

d) In the Newtonian limit, show that the radius becomes independent of mass when $n = 1$. Note that $n = 1$ is possible only when the system is far from an ideal Fermi gas; i.e., interactions are non-negligible. What is the value of the mass-independent radius?