중성자별의 구조에 대한 수치적 풀이

- Solution of TOV equation

김진호 (한국천문연구원)

별

주계열성

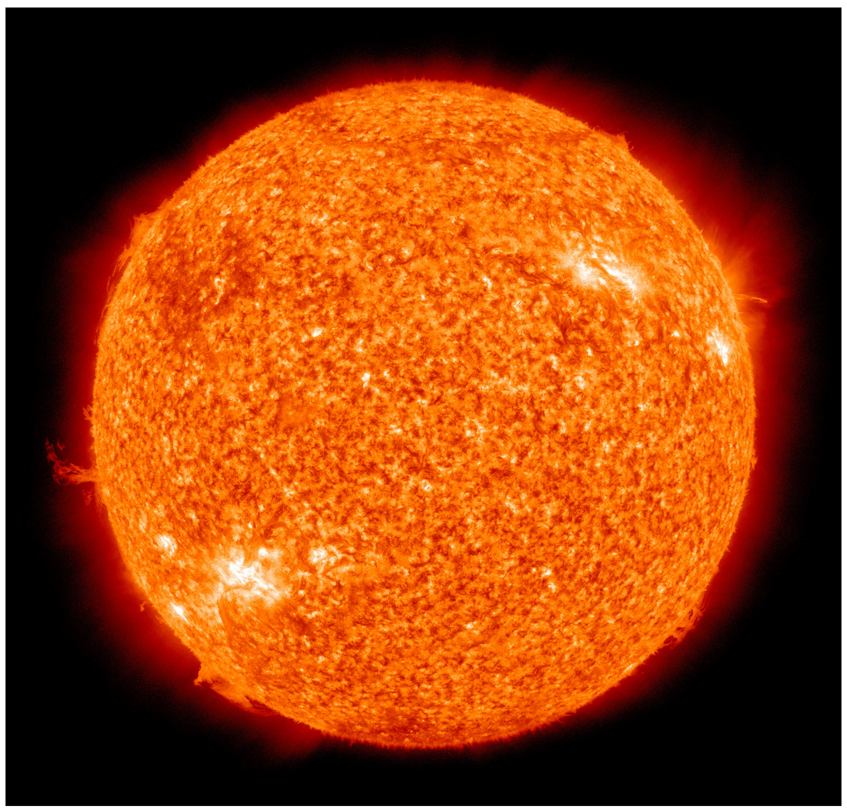
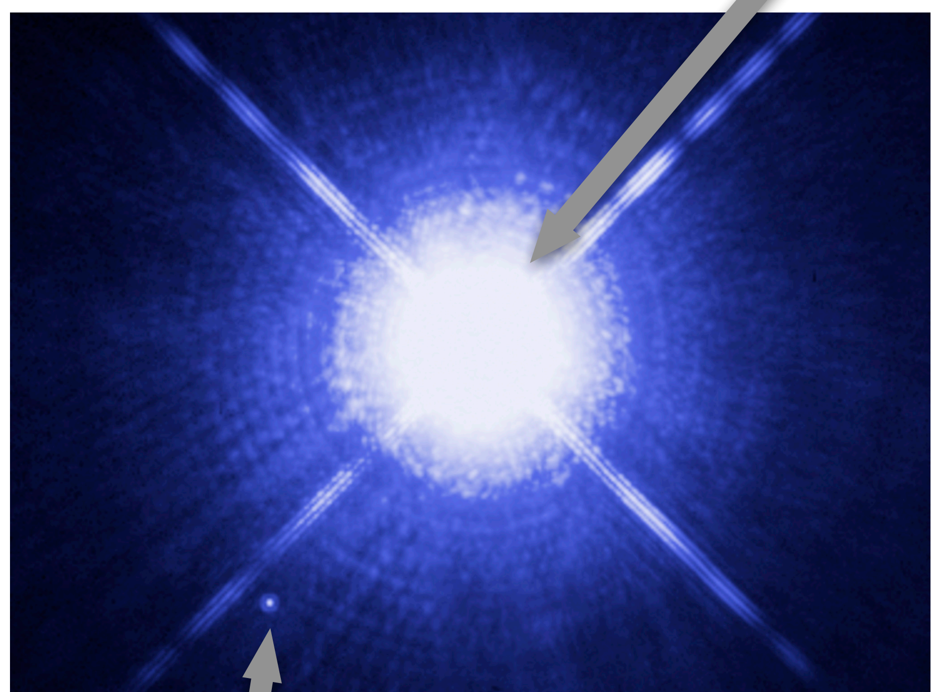


Image from NASA's
Solar Dynamics
Observatory

백색왜성

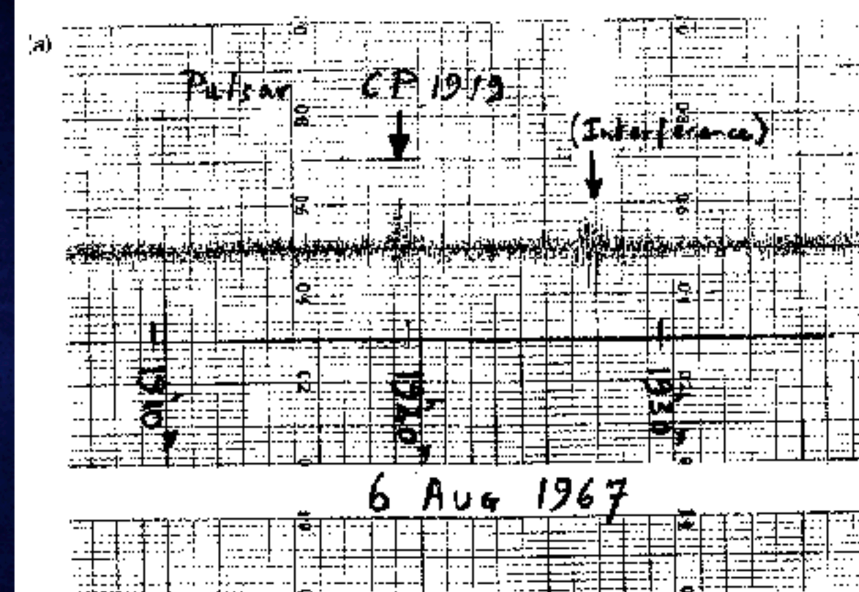
Sirius A



Sirius B

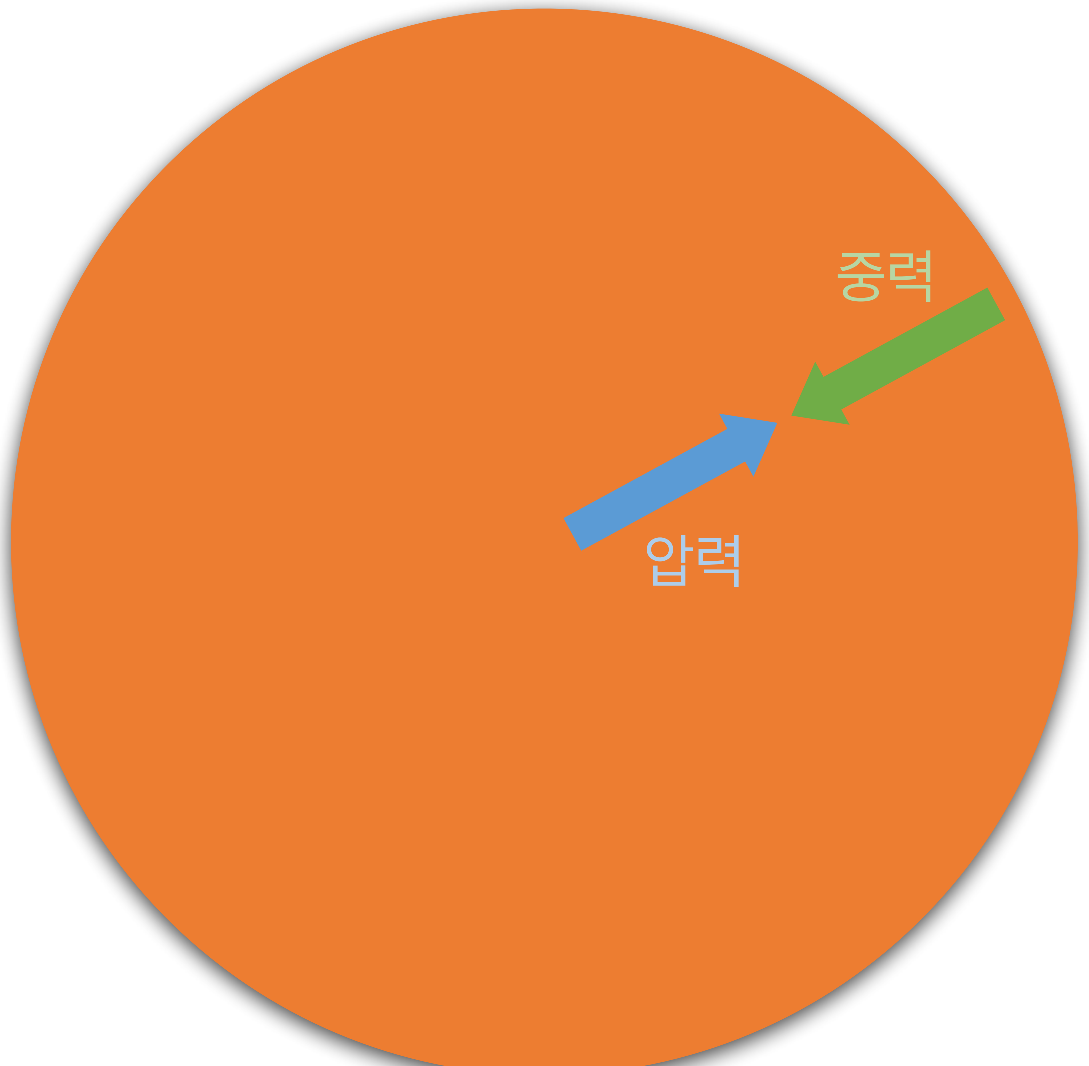
Image from HST

중성자별



Observation by Jocelyn Bell

별(항성)의 구조



항성

정역학적 평형 상태

압력

열압 (thermal): $P \sim T$

복사압 (radiation): $P \sim T^4$

축퇴압 (degenerate): $P \sim \rho^\alpha$

중력

뉴턴 중력

일반상대론

열압 vs 축퇴압

Thermal Pressure

characterized by kinetic motion

Degenerate Pressure

characterized by quantum states

Equation of State of White Dwarf

$$P = \frac{\pi m_e^4 c^5}{h^3}\left[x_F\left(1+x_F^2\right)^{\frac{1}{2}}\left(\frac{2}{3}x_F^2-1\right)+\ln\left[x_F+\left(1+x_F^2\right)^{\frac{1}{2}}\right]\right]$$

$$= \frac{8\pi m_e^4 c^5}{15h^3}\left[x_F^5-\frac{5}{14}x_F^7+\frac{5}{24}x_F^9+\cdots\right] \quad \text{for} \quad x_F \ll 1$$

: non-relativistic limit

$$= \frac{2\pi m_e^4 c^5}{3h^3}\left[x_F^4-x_F^2+\frac{3}{2}\ln\left(2x_F\right)+\cdots\right] \quad \text{for} \quad x_F \gg 1$$

: ultra-relativistic limit

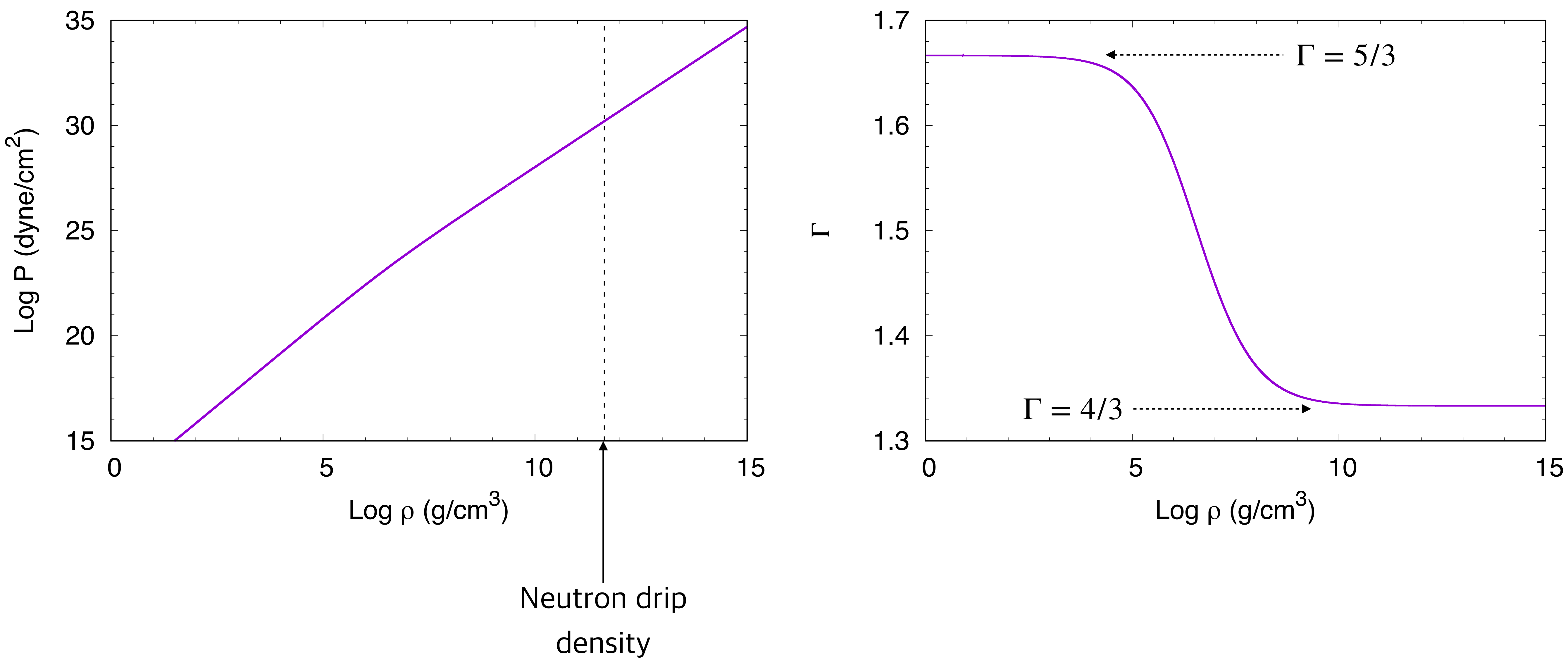
Recall that $p_F = m_e c x \sim n_e^{1/3} \sim \rho^{1/3}$.

Above asymptotic limit of the pressure gives polytropic equation of state i.e., $P = K\rho^{\Gamma}$.

$$\Gamma = \frac{5}{3} \text{ : non-relativistic}$$

$$\Gamma = \frac{4}{3} \text{ : relativistic}$$

Equation of State for White Dwarf



중성자별의 상태방정식

내일 (화) 이창환 교수님 수업 참고

Equilibrium Structure of Star

Fluid equation (Euler equation).

1. Continuity Equation

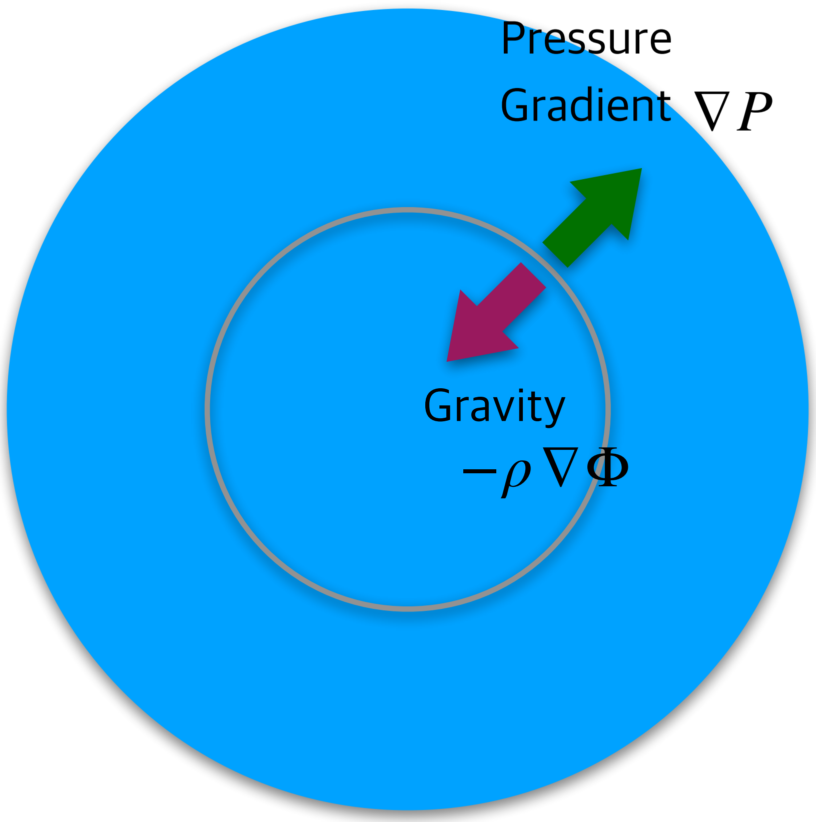
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v}\right) = 0$$

2. Momentum Equation

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} + \nabla P = -\rho \nabla \Phi$$

3. Energy Equation

$$\rho \frac{\partial e}{\partial t} + \vec{v} \cdot \nabla e + \frac{P}{\rho} \nabla \cdot \vec{v} = 0$$



For spherical star (non-rotating, non-magnetized), LHS of 1. Continuity and 3. Energy equation is 0. The remaining equation is 2. Momentum equation and can be rewritten in spherical coordinates as

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \quad \text{where } M_r = 4\pi \int_0^r \rho(r) r^2 dr \quad \Rightarrow \quad \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho$$

Lane-Emden Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Polytropic equation of state: $P = K\rho^{\Gamma} = K\rho^{1+1/N}$

This equation can further be reduced to dimensionless form by writing

$\rho = \rho_c\theta^N,\ \ r = a\xi,\ \ a = \left[\frac{(N+1)\,K\rho_c^{\frac{1}{N}-1}}{4\pi G}\right]^{1/2}$, where ρ_c is the central density.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^N$$

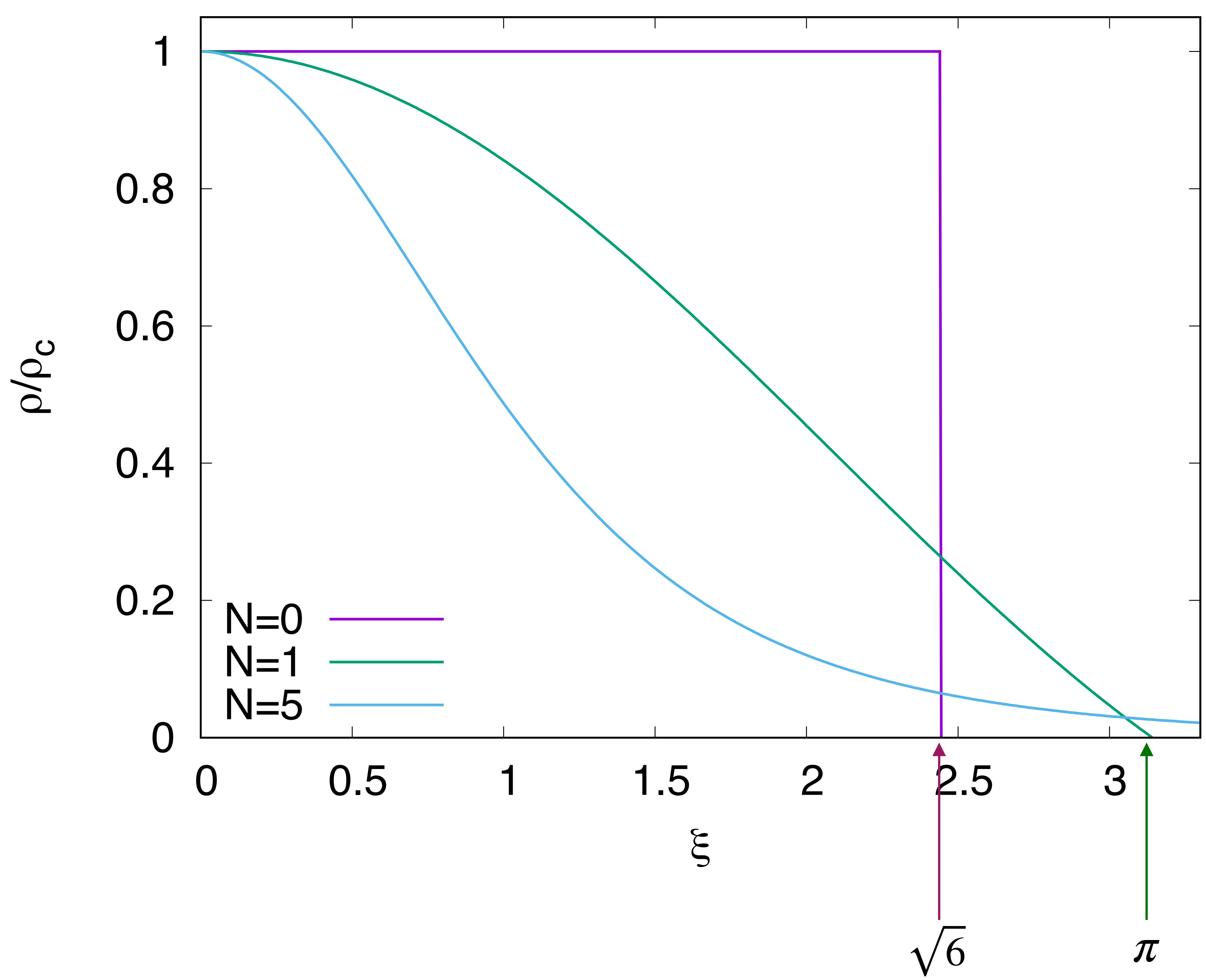
Lane-Emden Equation

Boundary condition: $\theta(0) = 1,\ \frac{d\theta(0)}{dr} = 0.$

Analytical solutions are available for particular N values (N=0, 1 and 5)

For N=0, $\theta(\xi) = 1 - \frac{1}{6}\xi^2$, For N=1, $\theta(\xi) = \frac{\sin\xi}{\xi}$, For N=5, $\theta(\xi) = \frac{1}{\sqrt{1+\xi^2/3}}$

Solution of Lane-Emden Equation



1. Conventionally, the equation of state is
   - hard (stiff) when Γ is large or N is small.
   - soft when Γ is small or N is large.

2. Density gradient with respect to ξ is
   - small for hard EoS.
   - large for soft EoS.

3. Recall that the EoS of relativistic degenerate gas (N=3) is softer than that of non-relativistic gas (N=1.5).
   - Relativistic degenerate gas can form more compact star than non-relativistic counterpart.

4. N=5 solution can extend to infinity while the total mass of the solution is finite.

Mass - Radius Relation

$$R = a\xi_s = \sqrt{\frac{(N+1)K}{4\pi G}}\rho_c^{\frac{1-N}{2N}}\xi_s$$

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_s} \xi^2 \theta^N dr$$

$$= -4\pi a^3 \rho_c \int_0^{\xi_s} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi$$

$$= -4\pi a^3 \rho_c \xi_s^2 \left|\theta'\left(\xi_s\right)\right|$$

$$= 4\pi \left[\frac{(N+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-N}{2N}} \xi_s^2 \left|\theta'\left(\xi_s\right)\right|$$

↓

$$M(R) = 4\pi R^{\frac{3-N}{1-N}} \left[\frac{(N+1)K}{4\pi G}\right]^{\frac{N}{N-1}} \xi_s^{\frac{5-3N}{1-N}} \left|\theta'\left(\xi_s\right)\right|$$

A star dominated by degenerate pressure becomes smaller as the mass of the star increases. This is the opposite of common sense that we generally know. ($1<N<3$).

Chandrasekhar Mass

As we see in the previous slide, total mass and radius of a star with polytropic EoS are independent on then central density when N=3 that corresponds to the relativistic Fermi electron gas.

$$R = 3.347 \times 10^4 \mathrm{km} \left(\frac{\rho_\mathrm{c}}{10^6 \mathrm{g/cm^{-3}}} \right)^{-1/3} \left(\frac{\mathrm{Y_e}}{0.5} \right)^{2/3}$$

$$M = M_\mathrm{ch} = 1.457 M_\odot \left(\frac{Y_e}{0.5} \right)^2$$

The Chandrasekhar limit occurs when it asymptotically approach to $\rho_c \to \infty$ which correspond to the relativistic environment. Resulting radius of the star is zero.

White dwarf with larger than Chandrasekhar mass cannot exist since a star with N>3 is unstable under radial pulsation. This fact will appear soon.

Relativistic Approach

To quantify how relativistic the object is, we can consider two dimensionless quantities as follows:

$$\xi = \frac{GM}{Rc^2}, \quad \beta = \frac{v}{c}.$$

Relativistic Objects

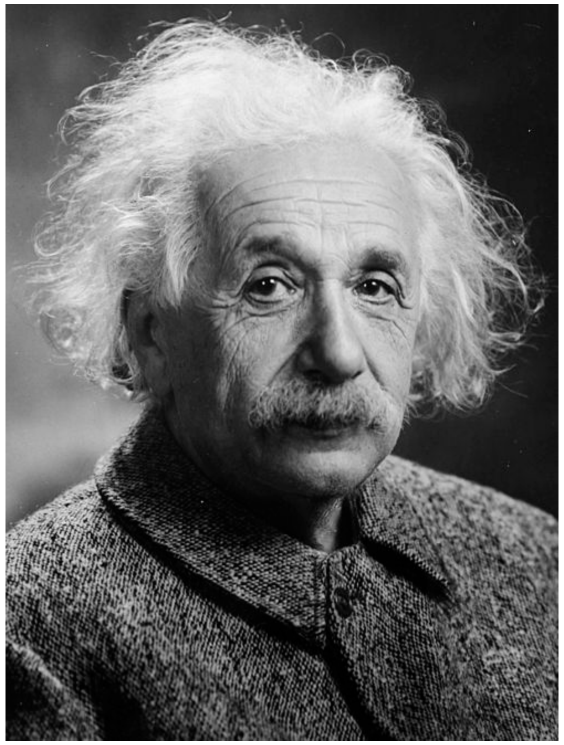
- Black hole: $R = 2GM/c^2$ (Schwarzschild BH), $R = GM/c^2$ (Extreme Kerr BH) -> $\xi = 0.5 \sim 1$.
- Neutron Star: $M \sim 1.4M_\odot$, $R \sim 10\mathrm{km}$ -> $\xi \sim 0.2$.

  $P_{\mathrm{rot}} \sim 1\mathrm{ms}$ -> $\beta \sim 0.2$.
- Jet: $\beta > 0.99$.

Newtonian Objects

- White Dwarf: $M \sim M_\oplus$, $R \sim R_\oplus$ -> $\xi \sim 0.0003$, $\beta \sim 0.0003$.
- Sun: $M \sim 1M_\odot$, $R \sim 1.4 \times 10^6\mathrm{km}$ -> $\xi \sim 10^{-6} \ll 1$.

General Relativity



- Albert Einstein proposed a new concept for studying the gravity.
- His idea was summarized in one sentence by John Wheeler.
- "Space-time tells matter how to move, matter tells space-time how to curve."
- In general relativity the stress energy tenser of perfect fluid and matter current are defined as

$$T^{ab} = \rho_0 h u^a u^b + P g^{ab},\ J^a = \rho_0 u^a,$$

where ρ_0, h and u^a are rest mass density, specific enthalpy and fluid 4-vector, respectively.

- Then the Einstein equation and fluid (Euler) equation in curved space-time can be written as

$$(1)\ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab},$$

$$(2)\ \nabla_a T^{ab} = 0,\ \nabla_a J^a = 0.$$

- Equation (1) shows that "matter (fluid) tells space-time how to curve".
- Equation (2) represents that "space-time tells matter (fluid) how to move".

Equilibrium Structure

- General relativistic counterpart of the Lane-Emden equation Tolmann-Oppenheimer-Volkoff (TOV) equation which is written as follows (Tolman 1939, Oppenheimer & Volkoff 1939)

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi r^3 P}{mc^2}\right)\left(1-\frac{2Gm}{rc^2}\right)^{-1},\ \frac{dM}{dr} = 4\pi r^2\rho.$$

where $\rho = \rho_0 + \rho_0\epsilon = \rho_0 h - P$. Here ρ contains all the energy sources (rest mass as well as internal energy).

- The space-time metric in this equation is

$$ds^2 = -e^{\nu}c^2dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where ν is determined by the constraint,

$$\frac{d\nu}{dr} = -\left(\frac{2}{P+\rho c^2}\right)\frac{dP}{dr}.$$

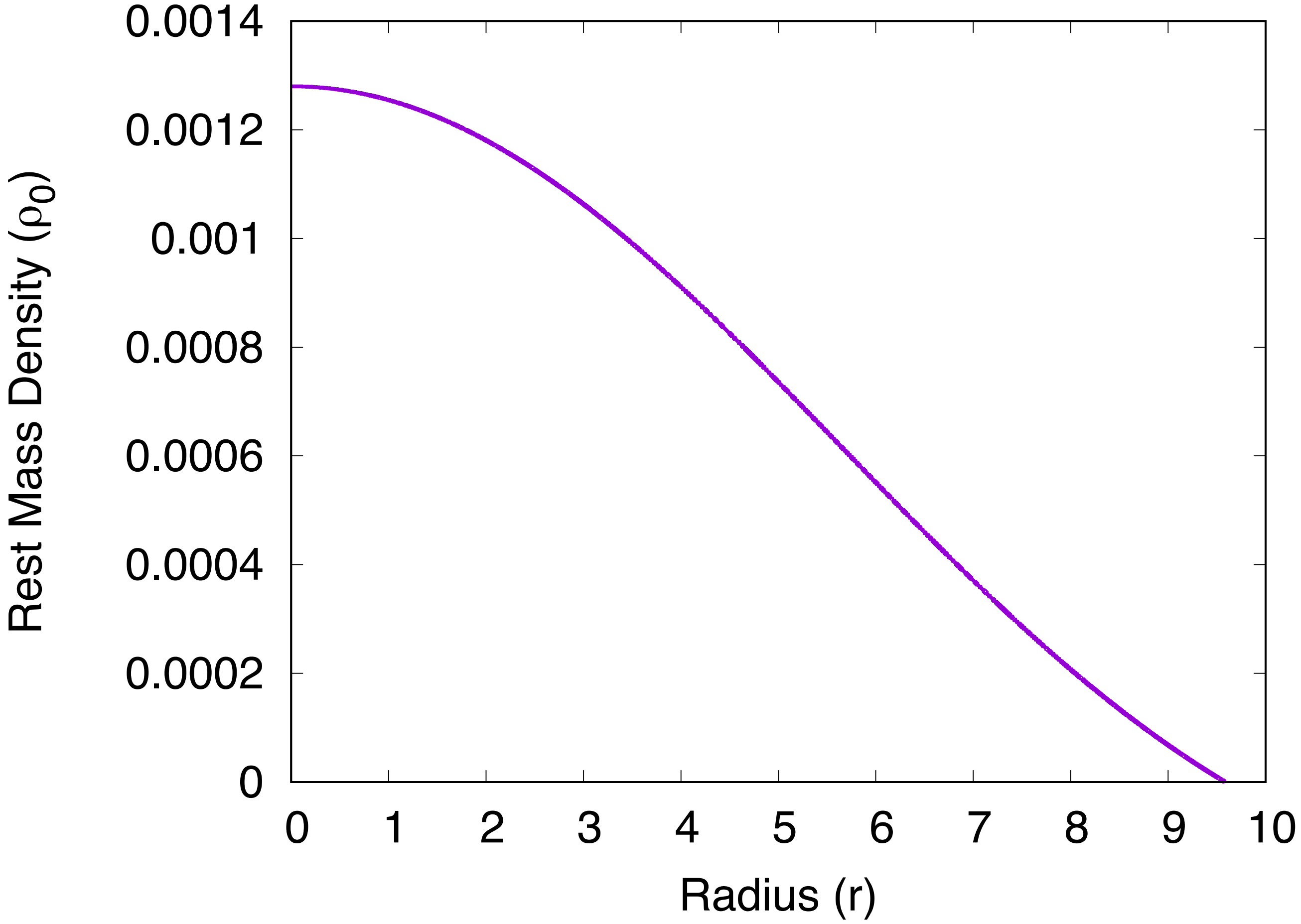
- Schwarzschild metric can be imposed for the boundary condition at the surface of the star.

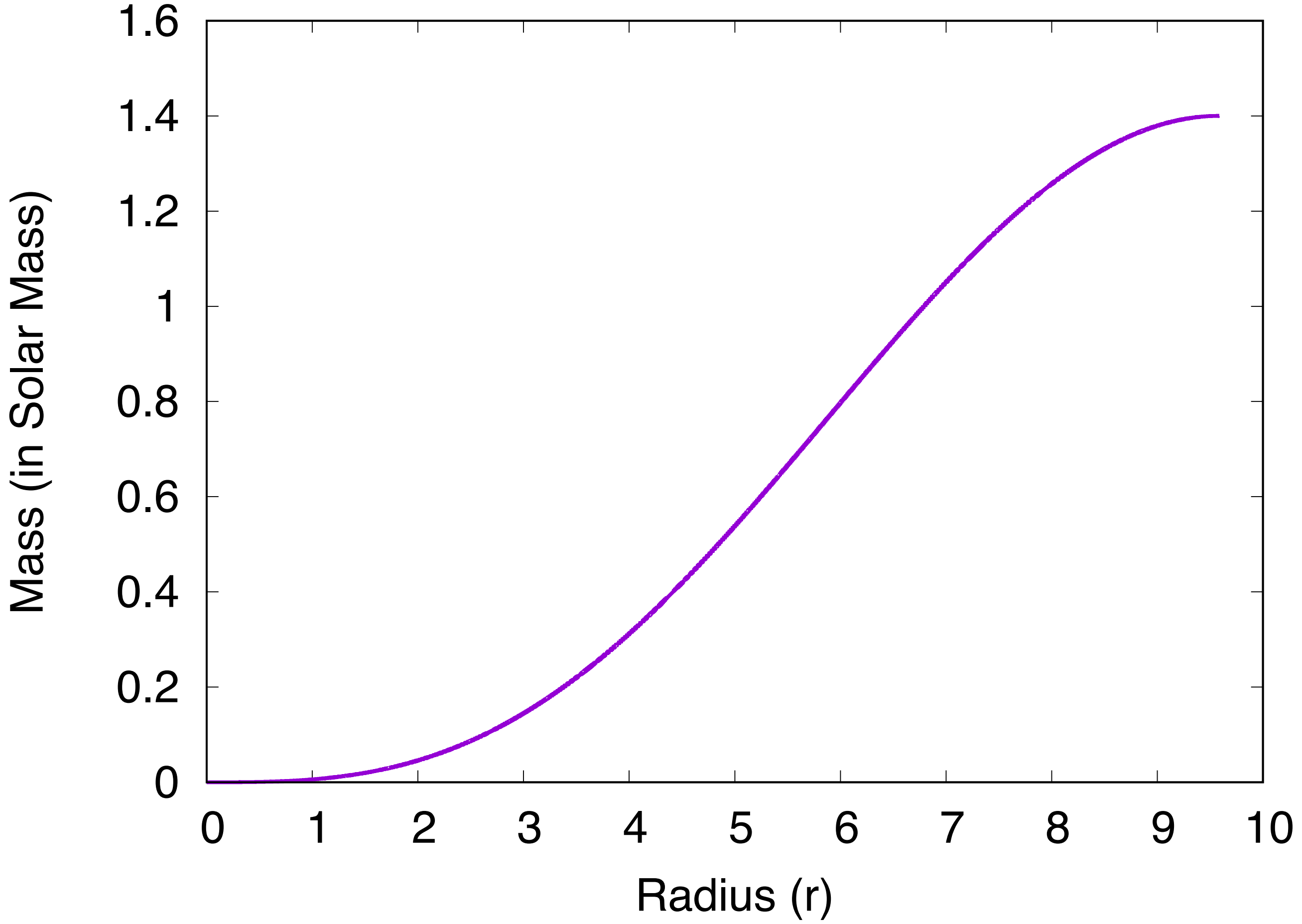
$$e^{\nu} = 1 - \frac{2GM}{rc^2}.$$

- Detailed derivation will not be covered in this lecture. Please read reference books if you want.

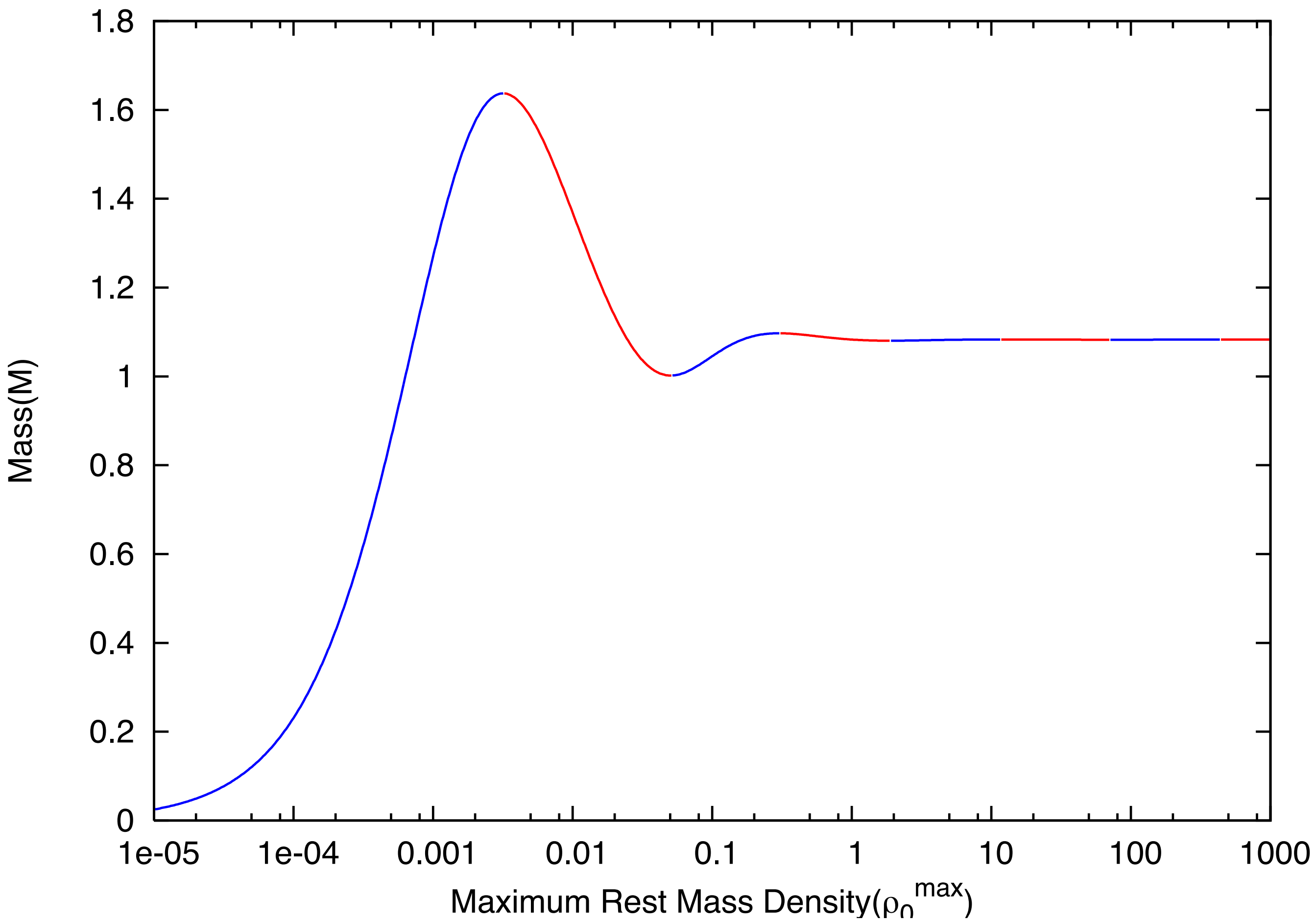
Solution of TOV Equation

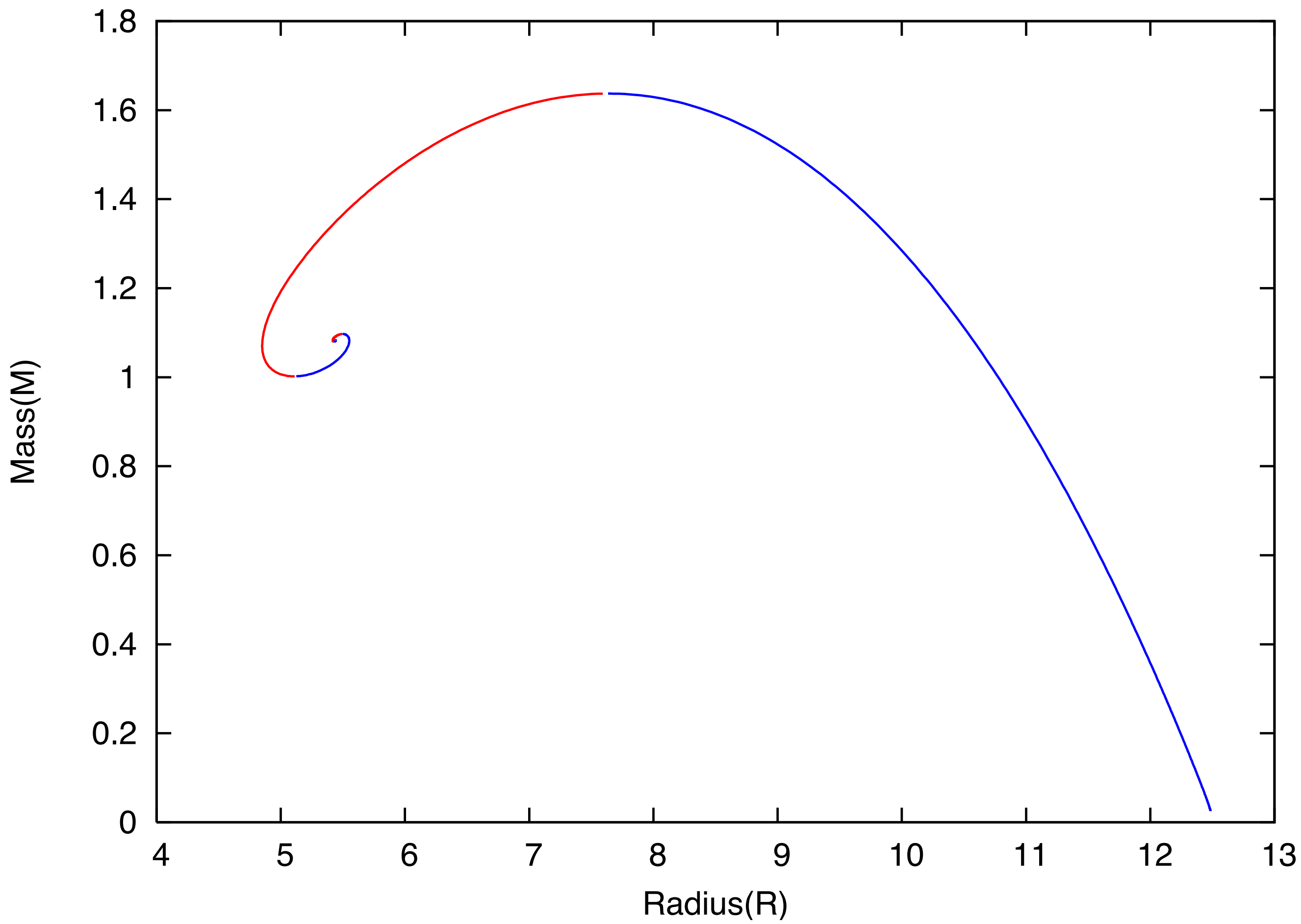
Solution of TOV equation using polytropic equation of state with N=1, K=100.





Sequence of Given EoS





- Same turning point method (even in relativistic calculation) can be applied to the neutron star.

$$\frac{dM}{d\rho_{\max}} > 0 \text{ or } \frac{dM}{dR} < 0\text{: Stable,}$$

$$\frac{dM}{d\rho_{\max}} < 0 \text{ or } \frac{dM}{dR} > 0\text{: Unstable.}$$

Realistic EoS

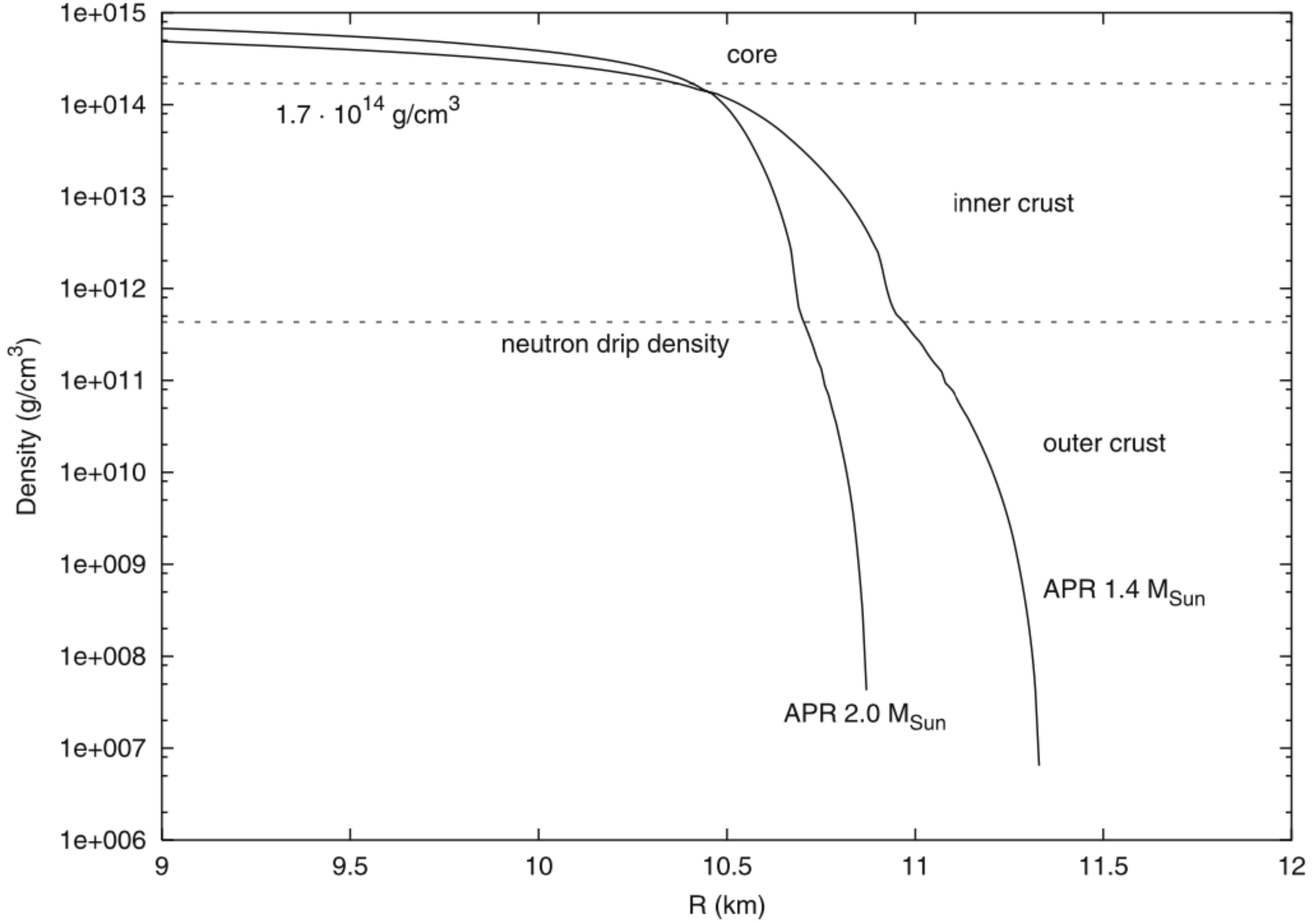
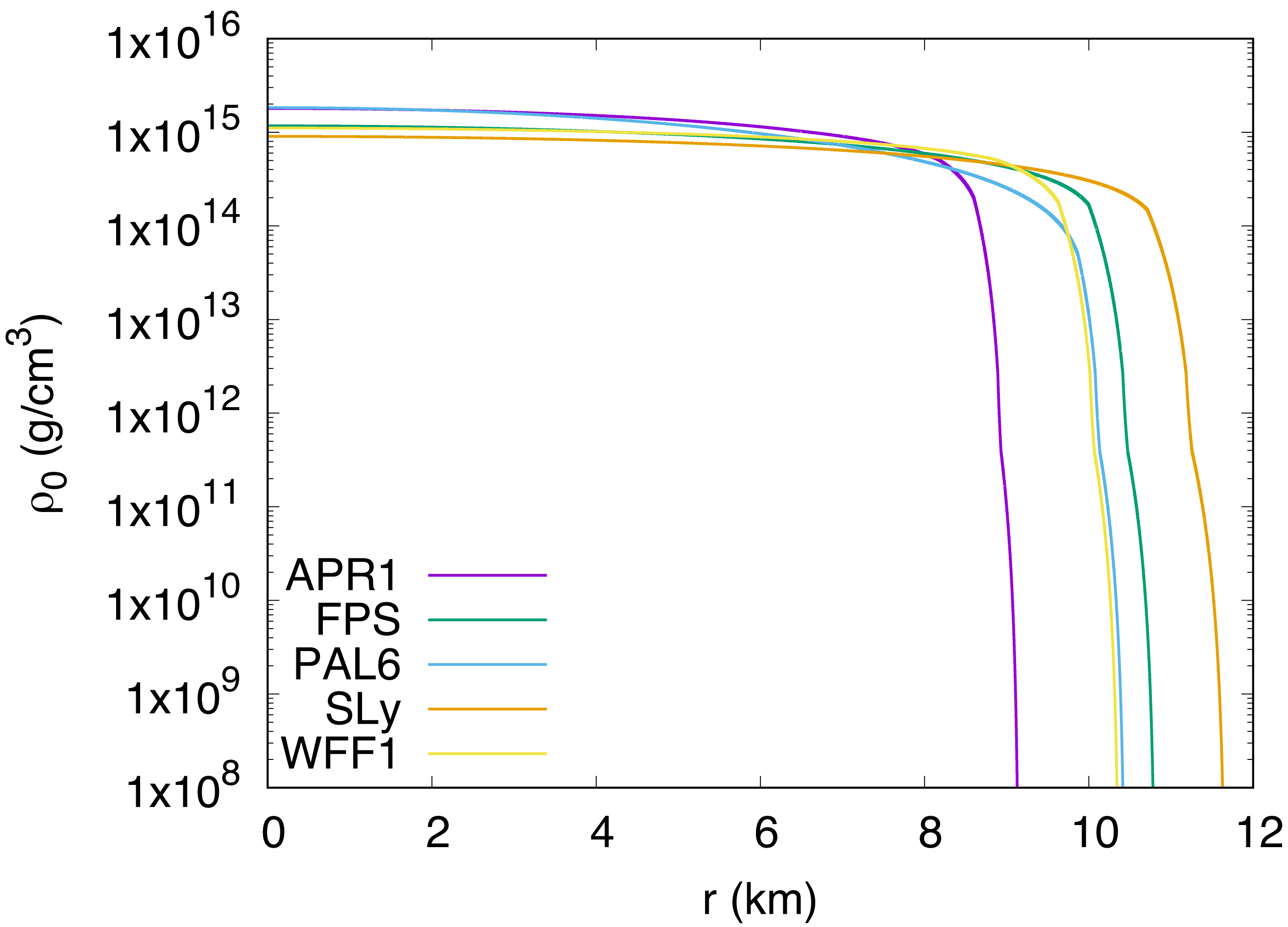


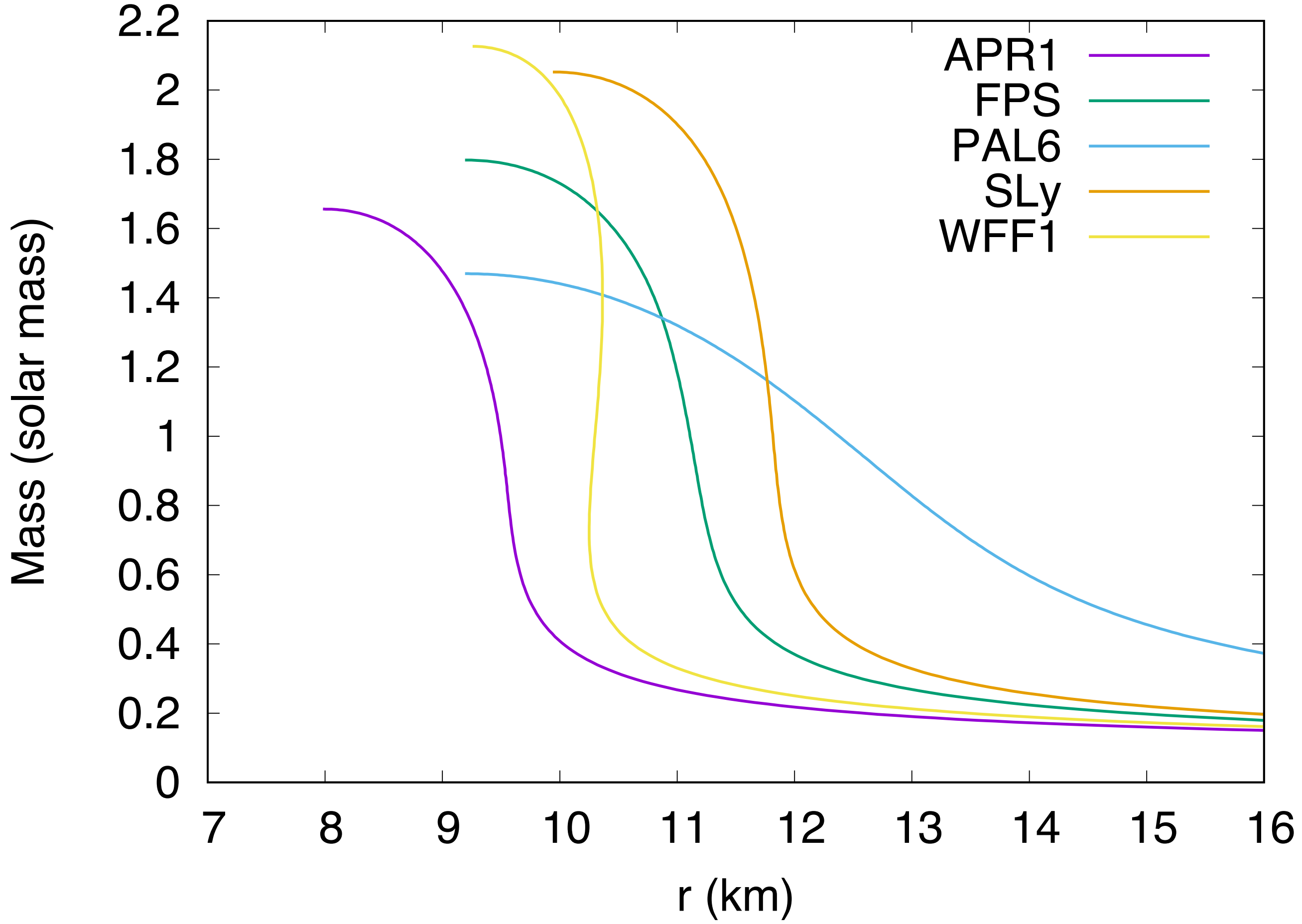
Image from Bauswein (2006)

Different EoS

1.4 solar mass neutron star with various EoSs



M-R relation



등방좌표계

Einstein field equation

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$$

$$dx^2 + dy^2 + dz^2$$

Metric assumption

$$ds^2 = -\,\mathrm{e}^{2\nu(\mathrm{r})}\mathrm{dt}^2 + \mathrm{e}^{2\lambda(\mathrm{r})}\left(\mathrm{dr}^2 + \mathrm{r}^2\mathrm{d}\theta^2 + \mathrm{r}^2\sin^2\theta\mathrm{d}\phi^2\right)$$

Non-zero Christoffel symbols

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \nu',$$

$$\Gamma^r_{tt} = \mathrm{e}^{2(\nu-\lambda)}\nu', \quad \Gamma^{\mathrm{r}}_{\mathrm{rr}} = \lambda', \quad \Gamma^{\mathrm{r}}_{\theta\theta} = -\,\mathrm{r}^2\lambda' - \mathrm{r}, \quad \Gamma^{\mathrm{r}}_{\phi\phi} = -\sin^2\theta\left(\mathrm{r}^2\lambda' + \mathrm{r}\right),$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \lambda' + \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \lambda' + \frac{1}{r}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

등방좌표계

Einstein field equation

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$$

Metric assumption

$$ds^2 = -\,\mathrm{e}^{2\nu(\mathrm{r})}\mathrm{dt}^2 + \mathrm{e}^{2\lambda(\mathrm{r})}\left(\mathrm{dr}^2 + \mathrm{r}^2\mathrm{d}\theta^2 + \mathrm{r}^2\sin^2\theta\,\mathrm{d}\phi^2\right)$$

Einstein tensor

$$G^t_t = \mathrm{e}^{-2\lambda}\left[(\lambda')^2 + 2\lambda'' + \frac{4}{\mathrm{r}}\lambda'\right],$$

$$G^r_r = \mathrm{e}^{-2\lambda}\left[2\nu'\lambda' + (\lambda')^2 + \frac{2}{\mathrm{r}}(\nu' + \lambda')\right],$$

$$G^\theta_\theta = G^\phi_\phi = \mathrm{e}^{-2\lambda}\left[(\nu')^2 + \nu'' + \lambda'' + \frac{1}{\mathrm{r}}(\nu' + \lambda')\right].$$

We will choose

$$G^t_t = \mathrm{e}^{-2\lambda}\left[(\lambda')^2 + 2\lambda'' + \frac{4}{\mathrm{r}}\lambda'\right]$$
$$= 8\pi T^t_t = -\,8\pi\left(\rho h - P\right),$$

$$G^i_i - G^t_t = 2\mathrm{e}^{-2\lambda}\left[\nu'' + (\nu')^2 + \nu'\lambda' + \frac{2}{\mathrm{r}}\nu'\right]$$
$$= 8\pi\left(T^i_i - T^t_t\right) = 8\pi\left(\rho h + 2P\right)$$

유체역학

Ideal fluid

$$T^{ab} = \rho h u^a u^b + P g^{ab}$$

$$\nabla_a T^{ab} = 0$$

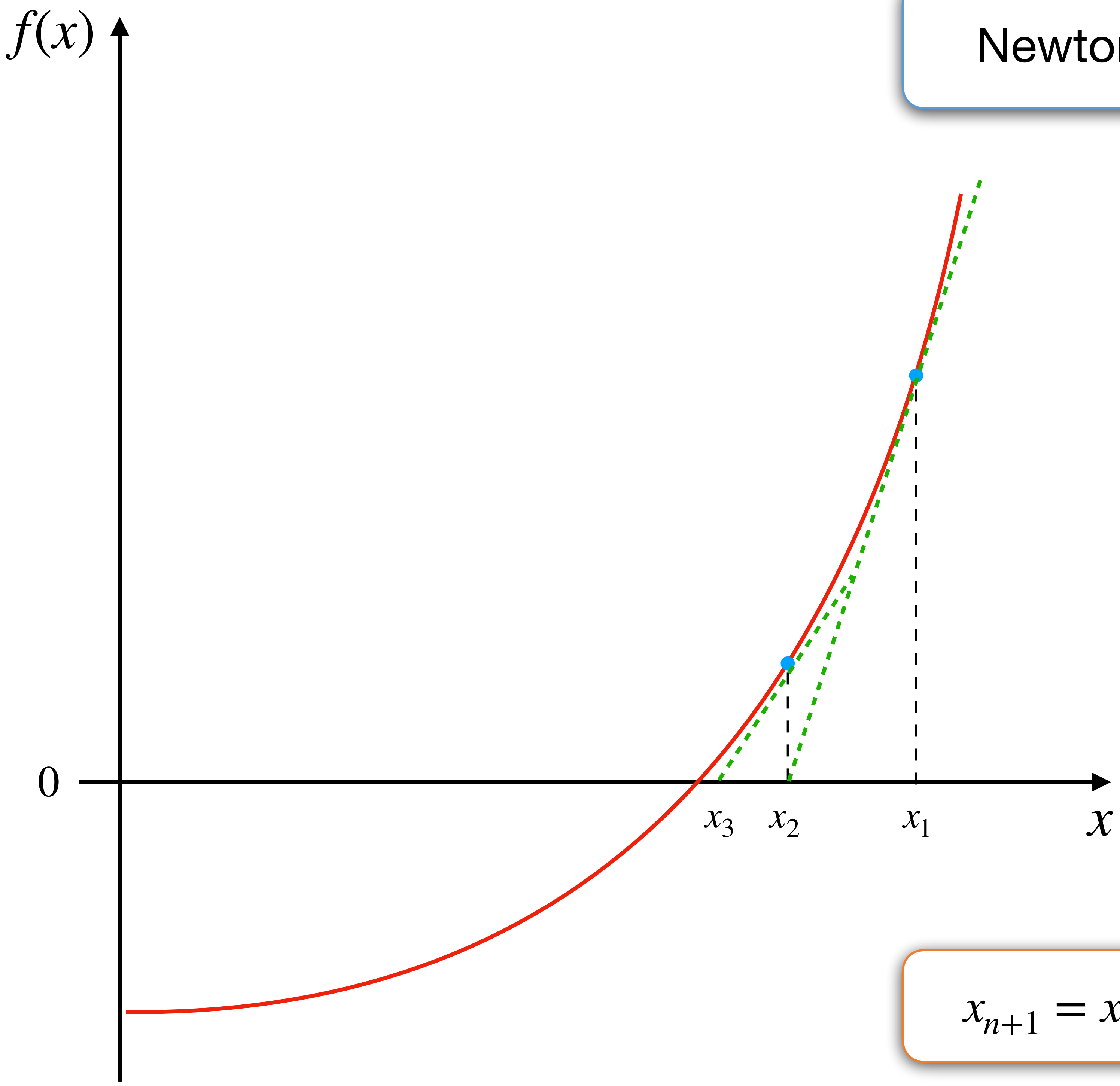
$$\gamma_r^b \nabla_a T_b^a$$
$$= \partial_a T_r^a + \Gamma_{ab}^a T_r^b - \Gamma_{ra}^b T_b^a$$
$$= \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} T_r^a \right) - \Gamma_{ra}^b T_b^a$$
$$= P' + \rho h \nu' = 0$$

$$\ln h + \nu = \text{const} .$$

수치적 방법

수치방법을 통한 방정식 풀이

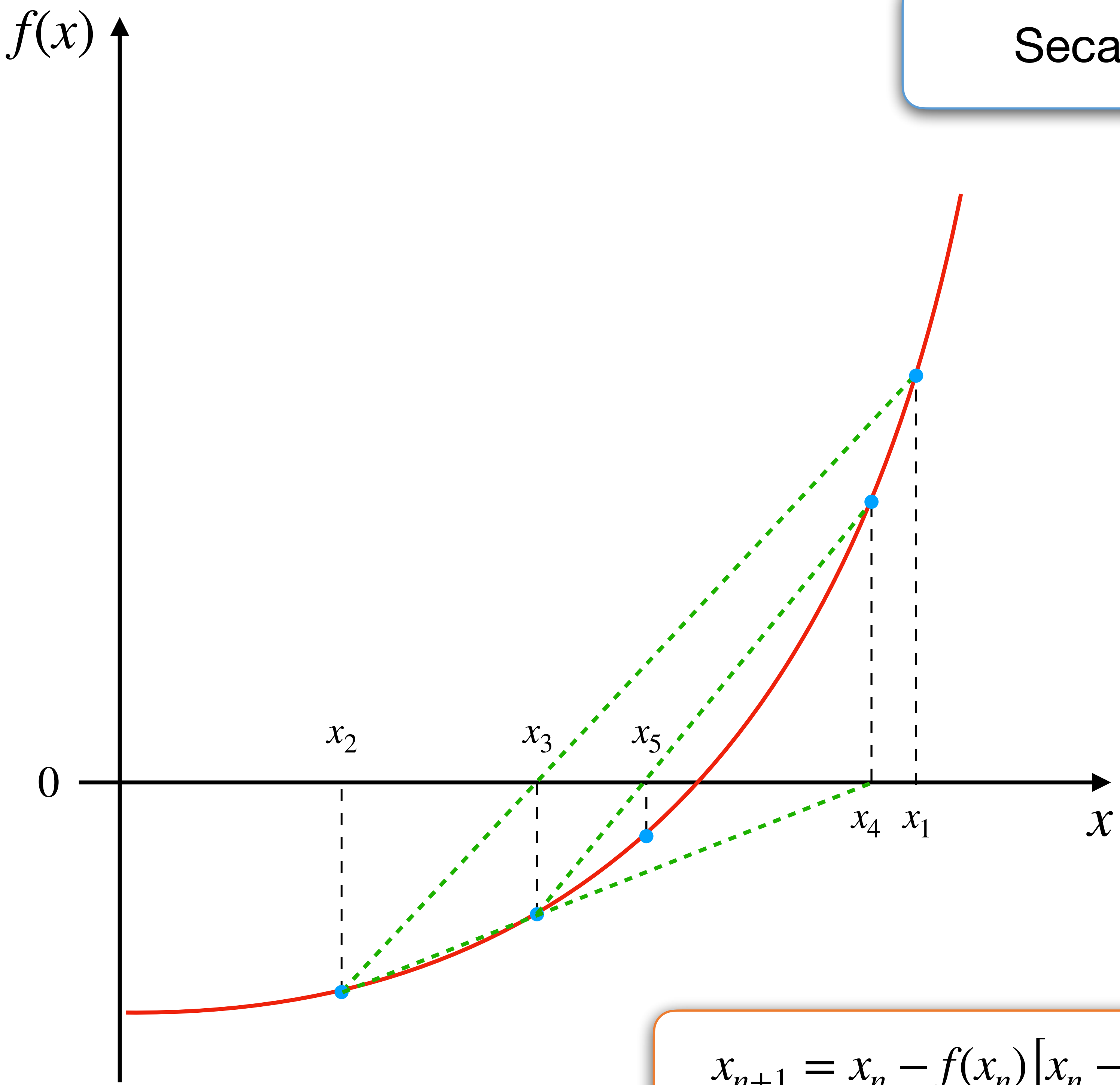
Newton - Raphson Method



$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

수치방법을 통한 방정식 풀이

Secant Method

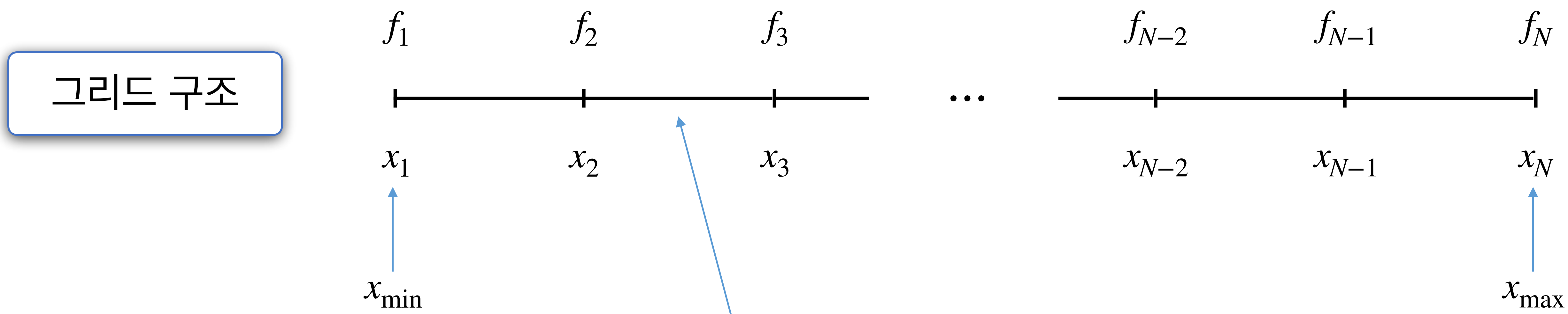


$$x_{n+1} = x_n - f(x_n)\left[x_n - x_{n-1}\right] / \left[f(x_n) - f(x_{n-1})\right]$$

할선법의 특징

- 할선법은 뉴턴 방법과 근접한 해의 수렴성을 갖는다 (bisection 방법에 비해 매우 빠름).
- Bisection 방법에 비해 잘못된 해를 줄 가능성이 높다.
- 할선법은 방정식의 미분 값을 알 수 없을때 유용하다.

유한 차분법



$$\Delta x = \frac{x_{\max} - x_{\min}}{N - 1}$$

유한 차분법

$$f(x_0+\Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2 + \frac{1}{6}f'''(x_0)\Delta x^3$$

$$f(x_0-\Delta x) = f(x_0) - f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2 - \frac{1}{6}f'''(x_0)\Delta x^3$$

$$f'(x_0) = \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x}$$

$$f''(x_0) = \frac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{\Delta x^2}$$

Centered differencing

유한 차분법

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2} f''(x_0)\Delta x^2 + \frac{1}{6} f'''(x_0)\Delta x^3$$

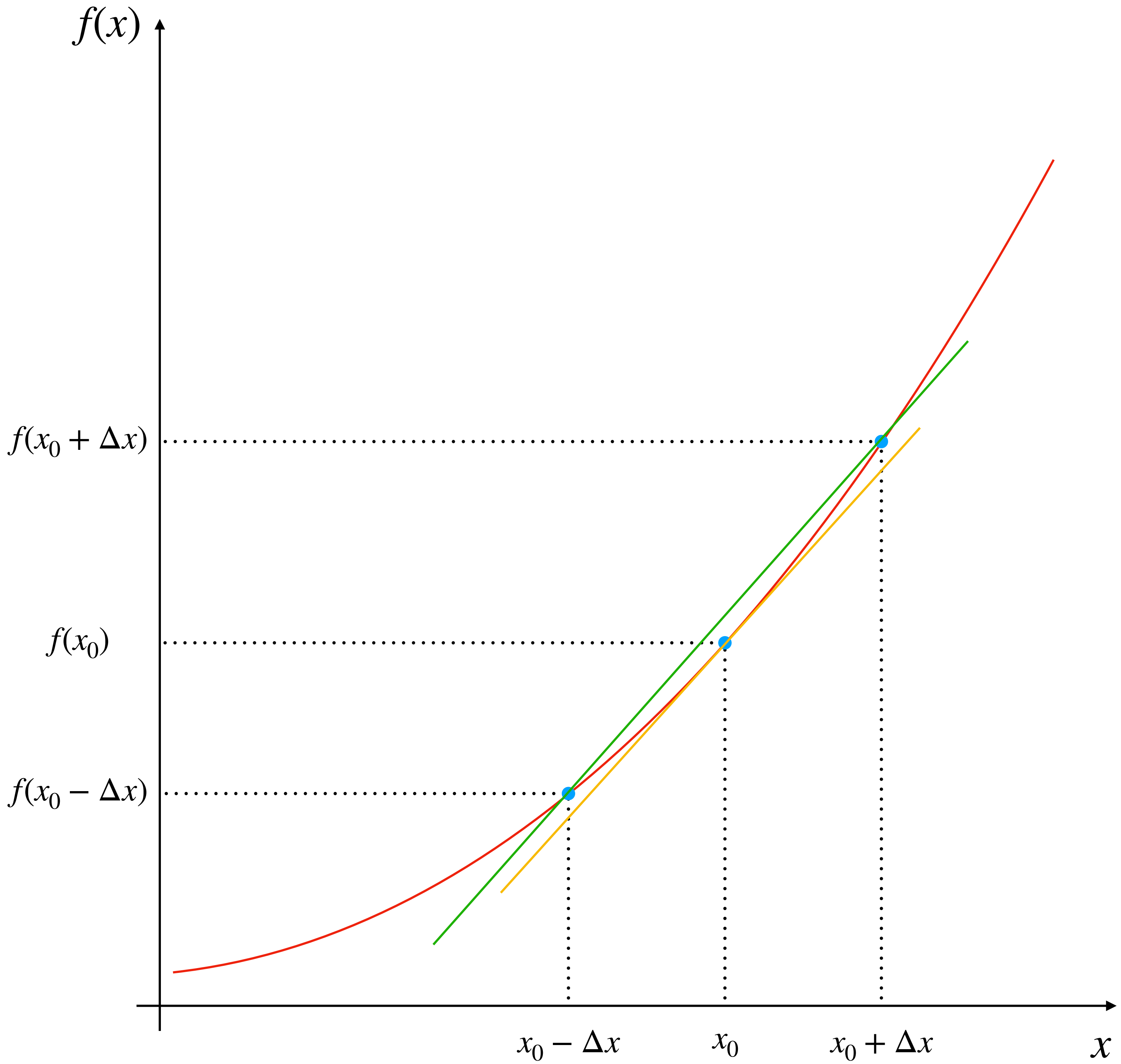
$$f(x_0 + 2\Delta x) = f(x_0) + 2f'(x_0)\Delta x + 2f''(x_0)\Delta x^2 + \frac{4}{3} f'''(x_0)\Delta x^3$$

$$f'(x_0) = \frac{-f(x_0 + 2\Delta x) + 4f(x_0 + \Delta x) - 3f(x_0)}{2\Delta x}$$

$$f''(x_0) = \frac{f(x_0 + 2\Delta x) - 2f(x_0 + \Delta x) + f(x_0)}{\Delta x^2}$$

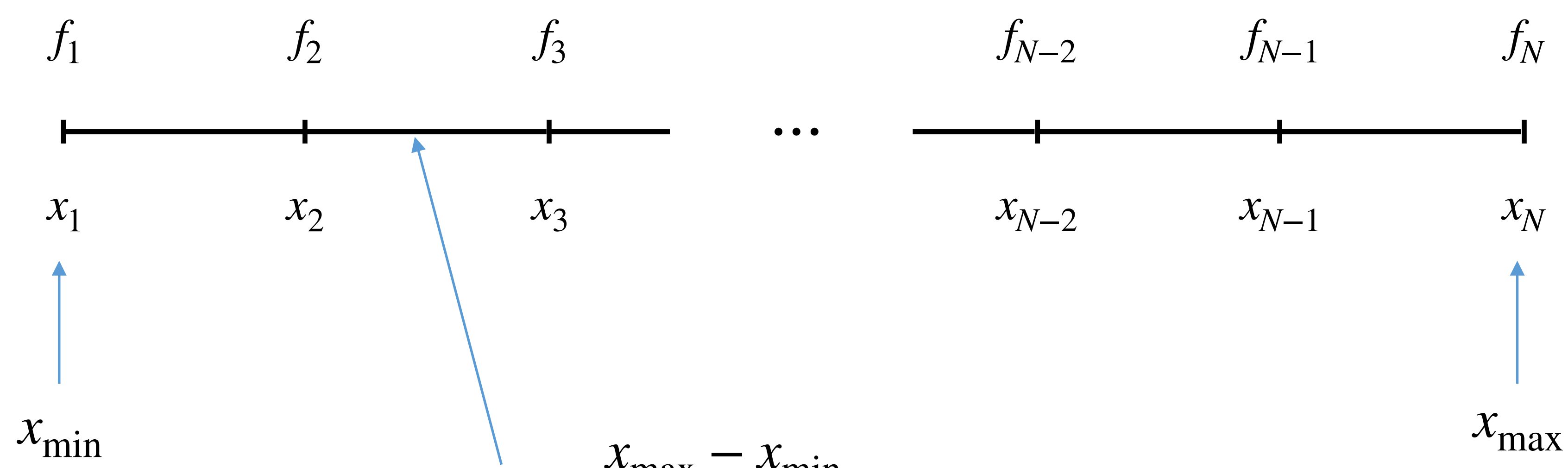
Forward differencing

유한 차분법



완화법

$$\frac{d^2 f}{dx^2} = g(x)$$



$$\Delta x = \frac{x_{\max} - x_{\min}}{N-1}$$

$$r_i = f_{i+1} - 2f_i + f_{i-1} - g_i \Delta x^2$$

$$f_i^{n+1} = f_i^n - r_i / \frac{\partial r_i}{\partial f_i}$$

완화법의 특징

- 완화법을 통한 풀이는 매우 안정적이다.
- 하지만 수렴성이 매우 느리다 (계산 속도가 매우 느리다).
- 그리드의 크기에 비례하는 에러는 빠르게 걸러주는 반면 전체적인 큰 변화는 매우 천천히 해에 수렴한다.
- 이 특성을 이용한 계층적인 그리드 구조를 가지는 계산을 통해 수렴성을 크게 개선할 수 있다 (Multigrid method).

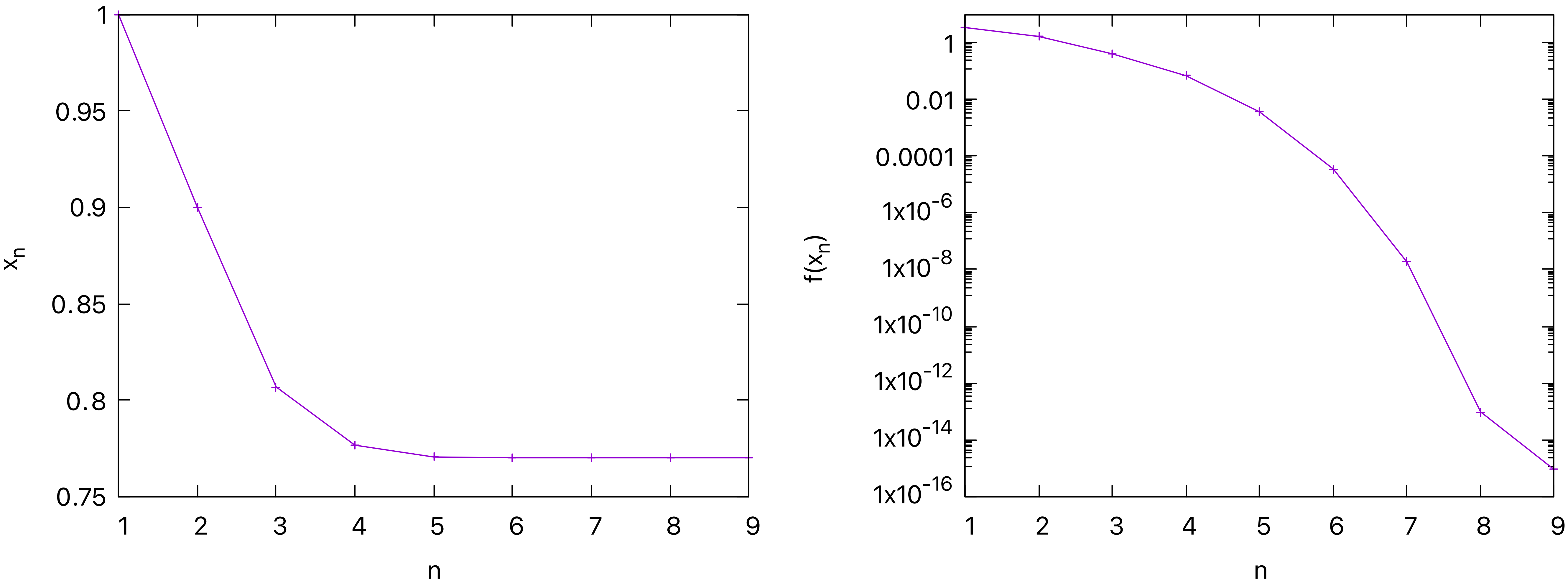
문제 1: 할선법을 이용한 비선형 방정식의 해 구하기

$$f(x) = 3 + x^2 - xe^{2x} \qquad x_1 = 1, \quad x_2 = 0.9$$

n	x_n	$f(x_n)$
1	1.0000000000000000E+00	-3.3890560989306504E+00
2	9.0000000000000002E-01	-1.6346827179716521E+00
3	8.0682241672647359E-01	-4.0015661075950071E-01
4	7.7662003728413431E-01	-6.7758198284421312E-02
	⋮	

$$\left| f(x_n) \right| < \text{tol} = 1 \times 10^{-14}$$

이 될 때까지 계산을 지속



x_n
1
0.95
0.9
0.85
0.8
0.75
1 2 3 4 5 6 7 8 9
n
f(x_n)
1
0.01
0.0001
1x10^-6
1x10^-8
1x10^-10
1x10^-12
1x10^-14
1x10^-16
1 2 3 4 5 6 7 8 9
n

문제 2: 유한 차분법과 완화법을 이용한 뿌아송 방정식 풀이

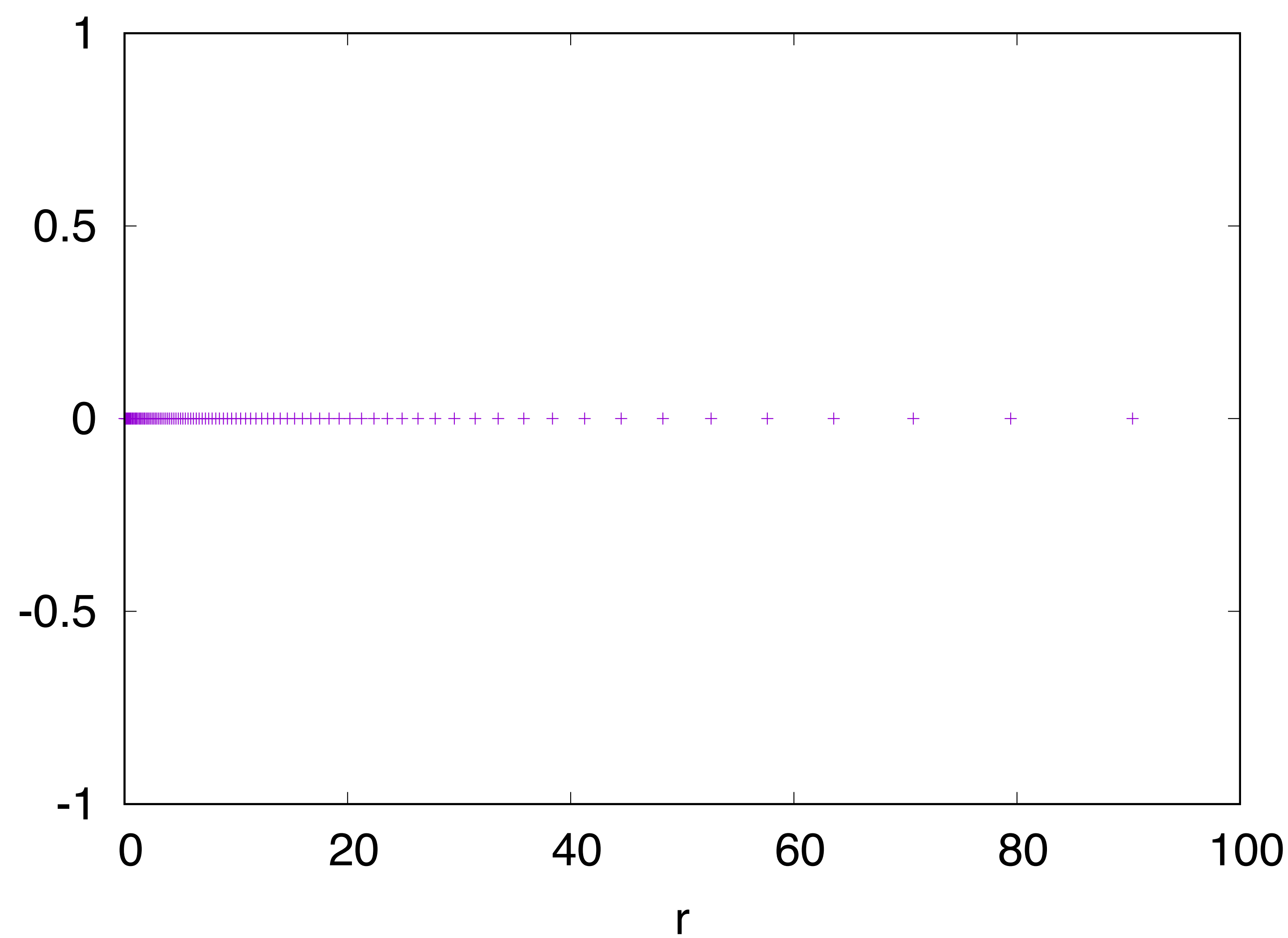
$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi\rho_0$$

좌표변환

$$\frac{d^2\Phi}{ds^2} + \frac{2}{s}\frac{d\Phi}{ds} = 4\pi\rho_0 r_s^2 (1-s)^{-4}$$

$$r = r_s\left(\frac{s}{1-s}\right)$$

$$s = 0 \rightarrow r = 0$$
$$s = 1/2 \rightarrow r = r_s$$
$$s = 1 \rightarrow r = \infty$$



$$\frac{d^2\Phi}{ds^2} + \frac{2}{s}\frac{d\Phi}{ds} = 4\pi\rho_0 r_s^2 (1-s)^{-4}$$

차분화

$$\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta s^2} + \frac{2}{s}\frac{\Phi_{i+1} - \Phi_{i-1}}{2\Delta s} = 4\pi\rho_0 r_s^2 (1-s)^{-4}$$

정리

$$r_i = \Phi_{i+1} - 2\Phi_i + \Phi_{i-1} + \frac{1}{s}\left(\Phi_{i+1} - \Phi_{i-1}\right)\Delta s - 4\pi\rho_0 r_s^2 (1-s)^{-4}\,\Delta s^2$$

경계조건

$s = 0 \ (r = 0)$ → $\frac{d\Phi}{ds} = 0$ → $\frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta s} = 0$

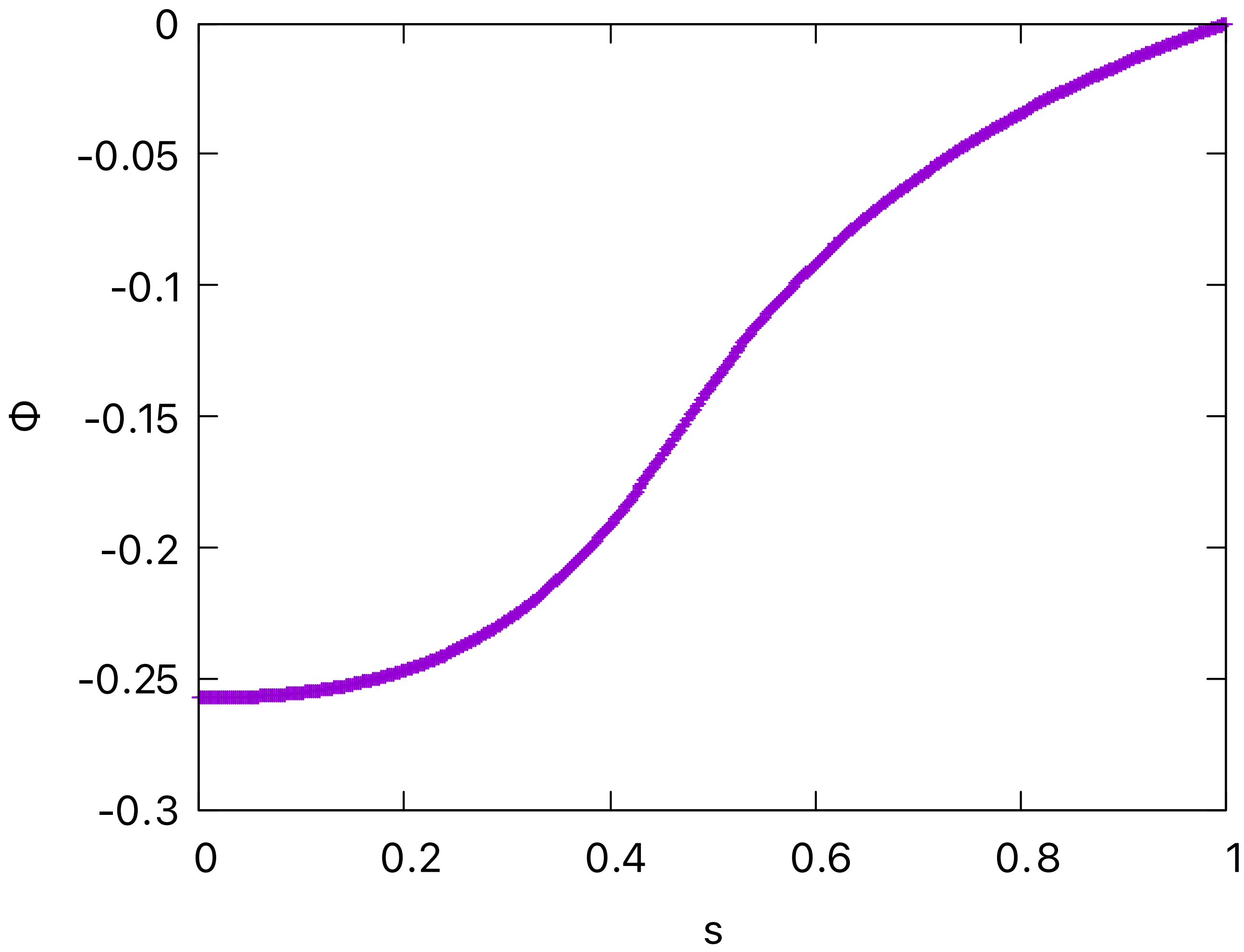
$s = 1 \ (r = \infty)$ → $\Phi = 0$

$n/10000$	L_1 norm of r_i	L_∞ norm of r_i	L_1 norm of difference (true - numerical)	L_∞ norm of difference (true - numerical)
1	5.0109932547092184E-06	1.6133744119961602E-05	1.1259869242093096E-01	2.1184500361626862E-01
2	3.8591442333689623E-06	1.2136732082335122E-05	9.0539141656252264E-02	1.6433087322862749E-01
3	2.9931845283822996E-06	8.7691916915710877E-06	7.3568761366881286E-02	1.2989197442695077E-01
4	2.3765451864331352E-06	6.5927485466410474E-06	6.0251564330984088E-02	1.0455402059800048E-01
5	1.9191721100935895E-06	5.1359617012636249E-06	4.9583982910048688E-02	8.5161077466815338E-02
6	1.5663280301966552E-06	4.0998301837902318E-06	4.0920979065003055E-02	6.9862119717043952E-02
7	1.2864450203351027E-06	3.3231354575891103E-06	3.3826902160134162E-02	5.7551665757009929E-02
8	1.0604616573810848E-06	2.7183906710082084E-06	2.7989006839431500E-02	4.7525131531390319E-02
9	8.7602641062685398E-07	2.2356701832038084E-06	2.3171178239209157E-02	3.9299937145575836E-02
10	7.2454471800758594E-07	1.8443851974581094E-06	1.9188684916534245E-02	3.2524217445069364E-02
11	5.9967123671666484E-07	1.5242951679761418E-06	1.5893621655597892E-02	2.6929100161045494E-02

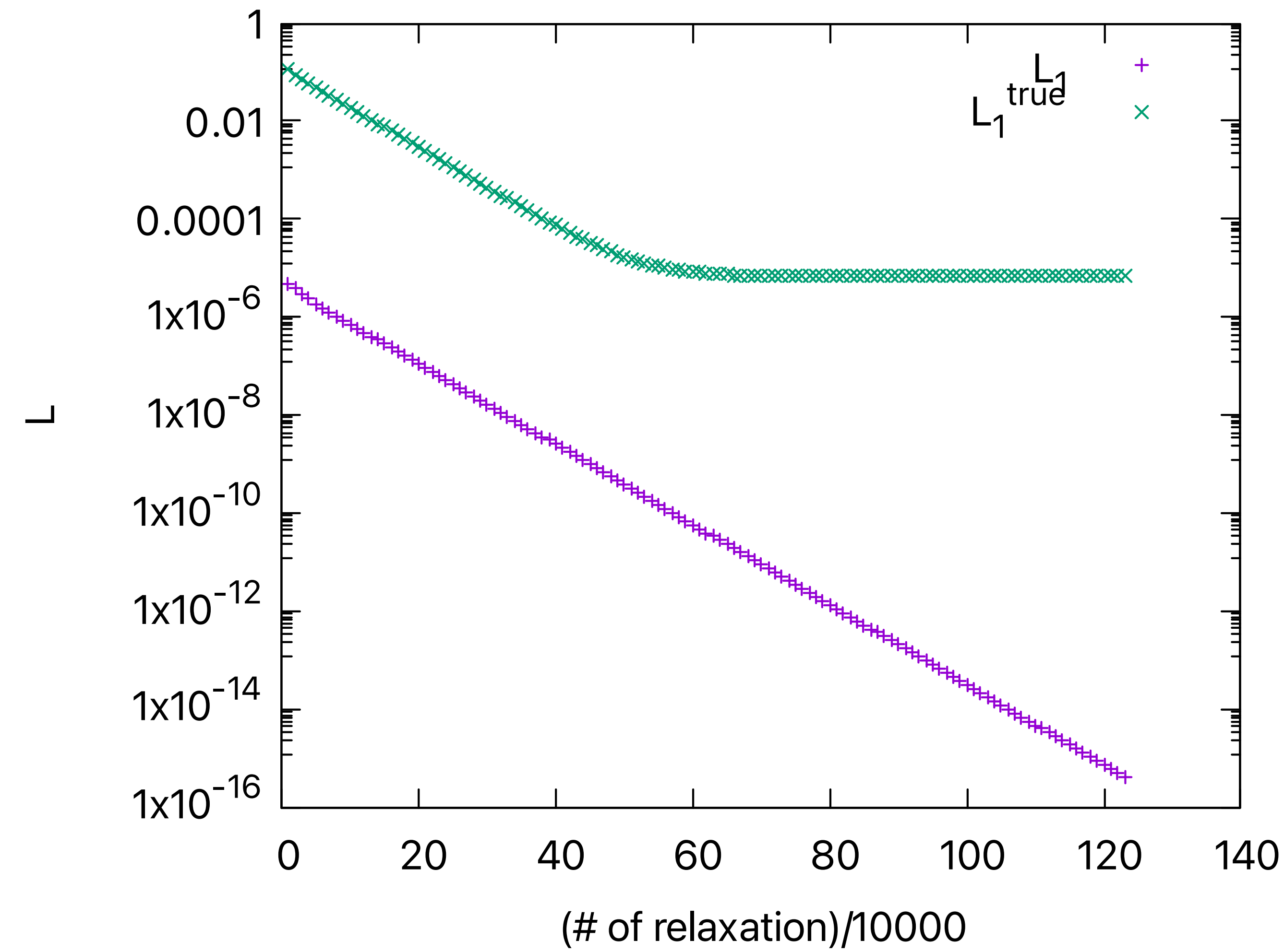
⋮

$L_\infty < \text{tol} = 1 \times 10^{-14}$
이 될 때까지 계산을 지속

최종결과



Convergence



문제 3: 주어진 밀도 분포로 인한 시공간의 휘어짐 구하기

$$e^{-2\lambda}\left[(\lambda')^2 + 2\lambda'' + \frac{4}{r}\lambda'\right] = -\,8\pi\left(\rho_0 h - P\right)$$

$$2e^{-2\lambda}\left[\nu'' + (\nu')^2 + \nu'\lambda' + \frac{2}{r}\nu'\right] = 8\pi\left(\rho_0 h + 2P\right)$$

좌표변환

$$r = r_s\left(\frac{s}{1-s}\right)$$

$$\frac{d^2\lambda}{ds^2} + \frac{2}{s}\frac{d\lambda}{ds} + \frac{1}{2}\left(\frac{d\lambda}{ds}\right)^2 = -\,4\pi e^{2\lambda} r_s^2\,(1-s)^{-4}\left(\rho_0 h - P\right),$$

$$\frac{d^2\nu}{ds^2} + \frac{2}{s}\frac{d\nu}{ds} + \left(\frac{d\nu}{ds}\right)^2 + \left(\frac{d\nu}{ds}\right)\left(\frac{d\lambda}{ds}\right) = 4\pi e^{2\lambda} r_s^2\,(1-s)^{-4}\left(\rho_0 h + 2P\right)$$

차분화 및 경계조건은 문제 2와 동일한 방법 적용

밀도 분포는 s좌표를 기준으로 포물선 프로파일 적용 (문제 2와 차이가 있음)

해석적 해가 존재하지 않음으로 r_i 기준으로 계산 종료 시점 판단

λ에 대한 방정식은 λ 만의 함수이므로 λ를 먼저 구한 후 이를 이용하여 ν를 구함

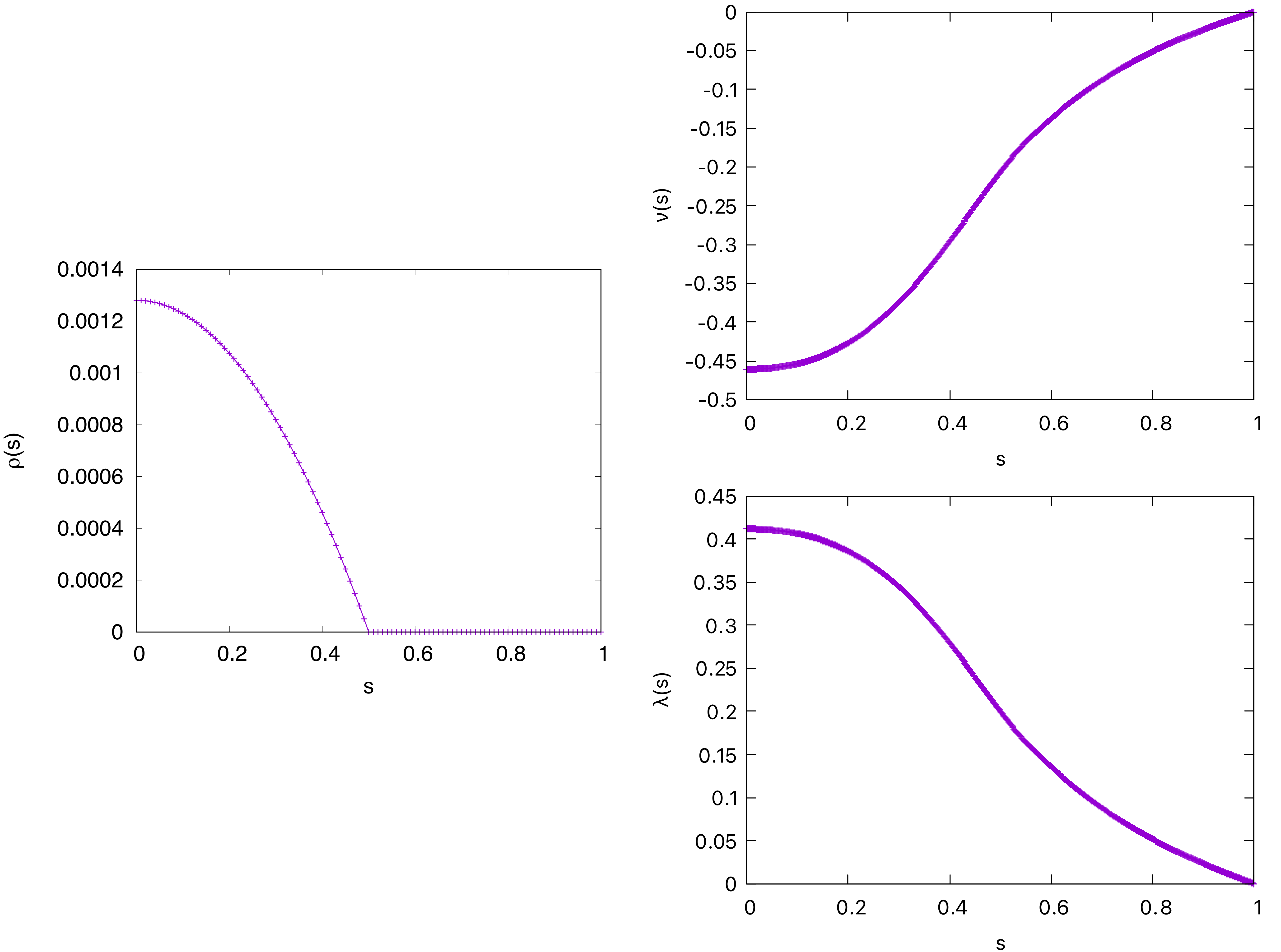
$n/10000$	L_1 norm of r_i	L_∞ norm of r_i
1	5.0799548586573619E-06	9.8803576553951888E-06
2	3.2202696213533162E-06	5.9347481291827098E-06
3	2.2089399002408043E-06	4.0689724747910994E-06
4	1.5713587571708805E-06	2.9061377953554235E-06
5	1.1462630179799290E-06	2.1274400670012739E-06
6	8.5209658794172813E-07	1.5858618482855746E-06
7	6.4263290284324557E-07	1.1986002645403282E-06
8	4.9008880395815977E-07	9.1561637211023950E-07
9	3.7700054629666103E-07	7.0525861817216651E-07
10	2.9196987825495831E-07	5.4675004212967337E-07
	⋮	

Residual of λ equation

$n/10000$	L_1 norm of r_i	L_∞ norm of r_i
1	7.7899537371440152E-06	1.4640843484403643E-05
2	3.2301892172685863E-06	5.5800749306222919E-06
3	1.3846650795534565E-06	2.3843111986443510E-06
4	5.8858647176740414E-07	1.0161815761633441E-06
5	2.4864667721528005E-07	4.2993841503635721E-07
6	1.0473556145890568E-07	1.8122387507313320E-07
7	4.4061685406107786E-08	7.6262235804147593E-08
8	1.8526672371964029E-08	3.2070042843557189E-08
9	7.7881907994528742E-09	1.3482216509874689E-08
10	3.2736698440460150E-09	5.6672065373852831E-09
	⋮	

Residual of ν equation

$L_\infty < \mathrm{tol} = 1 \times 10^{-14}$
이 될 때까지 계산을 지속



ρ(s)
0.0014
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
0
0.2
0.4
0.6
0.8
1
s
v(s)
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
-0.45
-0.5
0
0.2
0.4
0.6
0.8
1
s
λ(s)
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.2
0.4
0.6
0.8
1
s

문제 4: 할선법을 이용한 중성자별 크기 구하기

$$\nabla_a T^a_r = \frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}T^a_r\right) - \Gamma^c_{ar}T^a_c$$
$$= P' + \rho_0 h\nu' = 0$$

Integral representation

$$\ln h + \nu = C$$

$\ln h + \nu = C$ ← Integral representation

경계조건

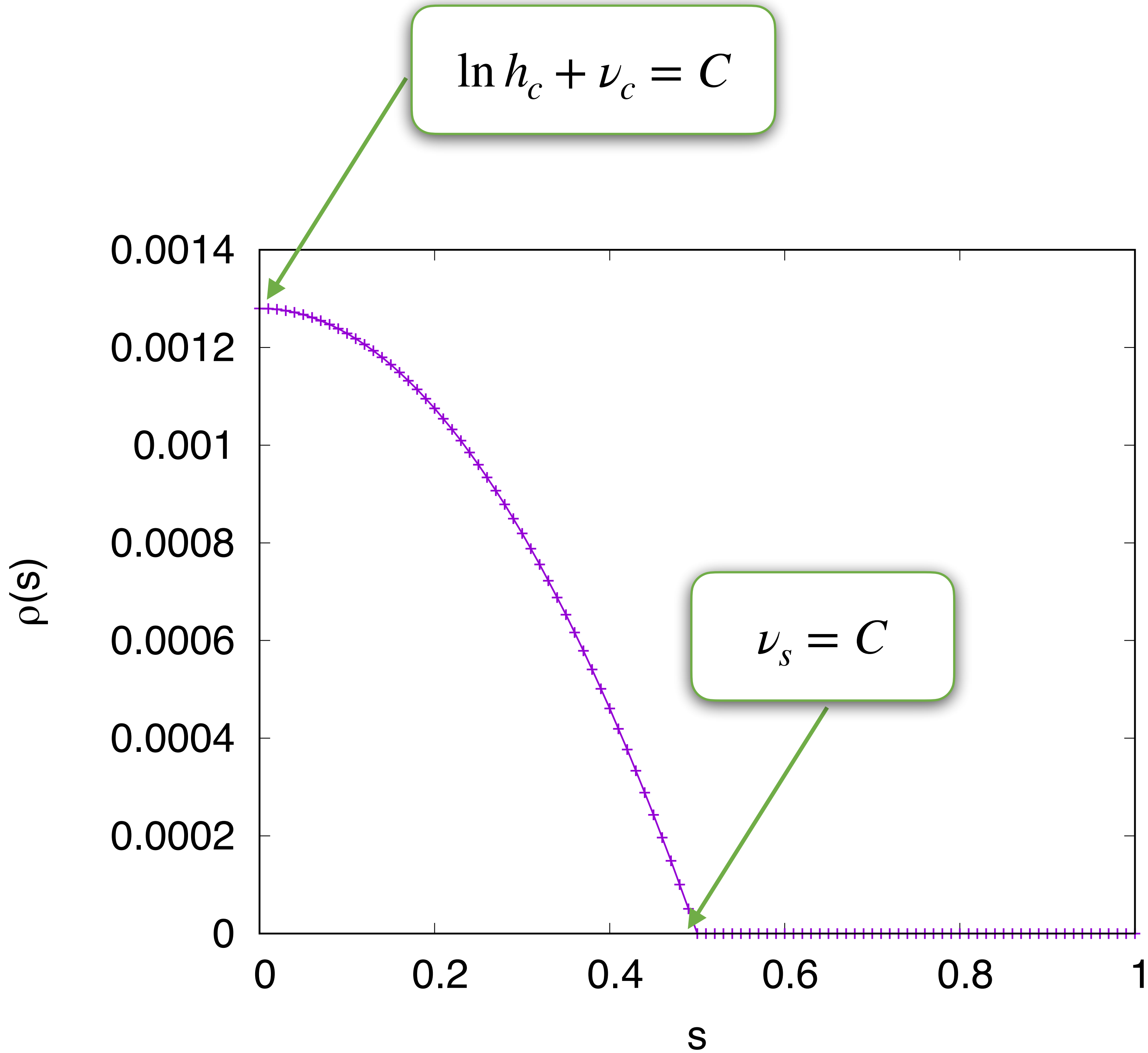
↓

$r_s, \quad C$

$f = \ln h_c + \nu_c - \nu_s$

↓

r_s^2



n $\quad$ r_s^2 $\quad$ f_n

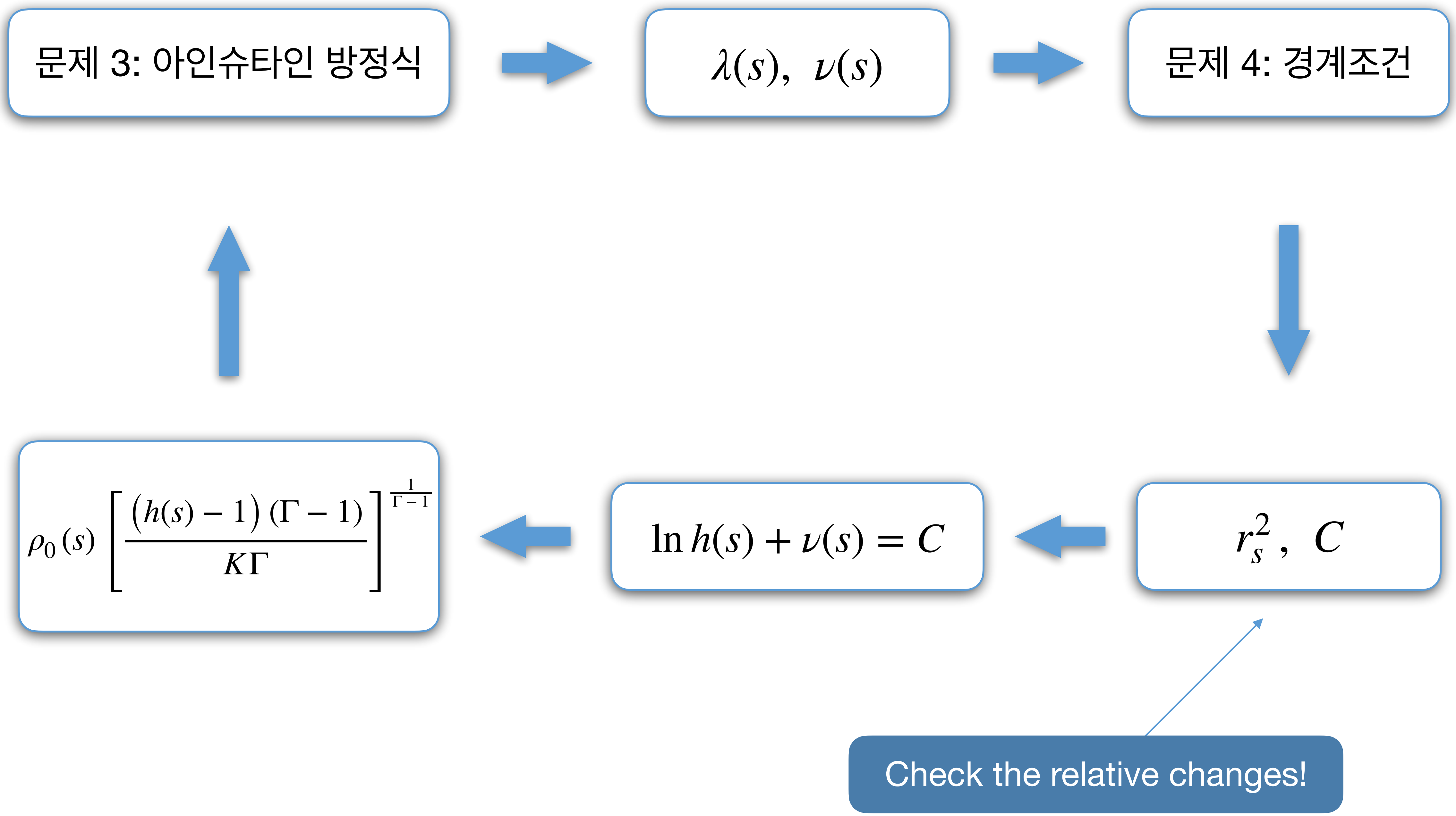
```
1    6.4000000000000000E+01    -2.6521504560154285E-02
2    6.3359999999999999E+01    -1.8229607860883512E-02
3    6.1952969804846759E+01    -2.4113176571275707E-03
4    6.1738484123399402E+01    -2.2496171238187146E-04
5    6.1716414951041742E+01    -2.8505992679950243E-06
                          ⋮
```

$$\left|f_n\right| < \mathrm{tol} = 1 \times 10^{-10}$$

이 될 때까지 계산을 지속

문제 5: 반복을 통한 최종 중성자별 구조 구하기

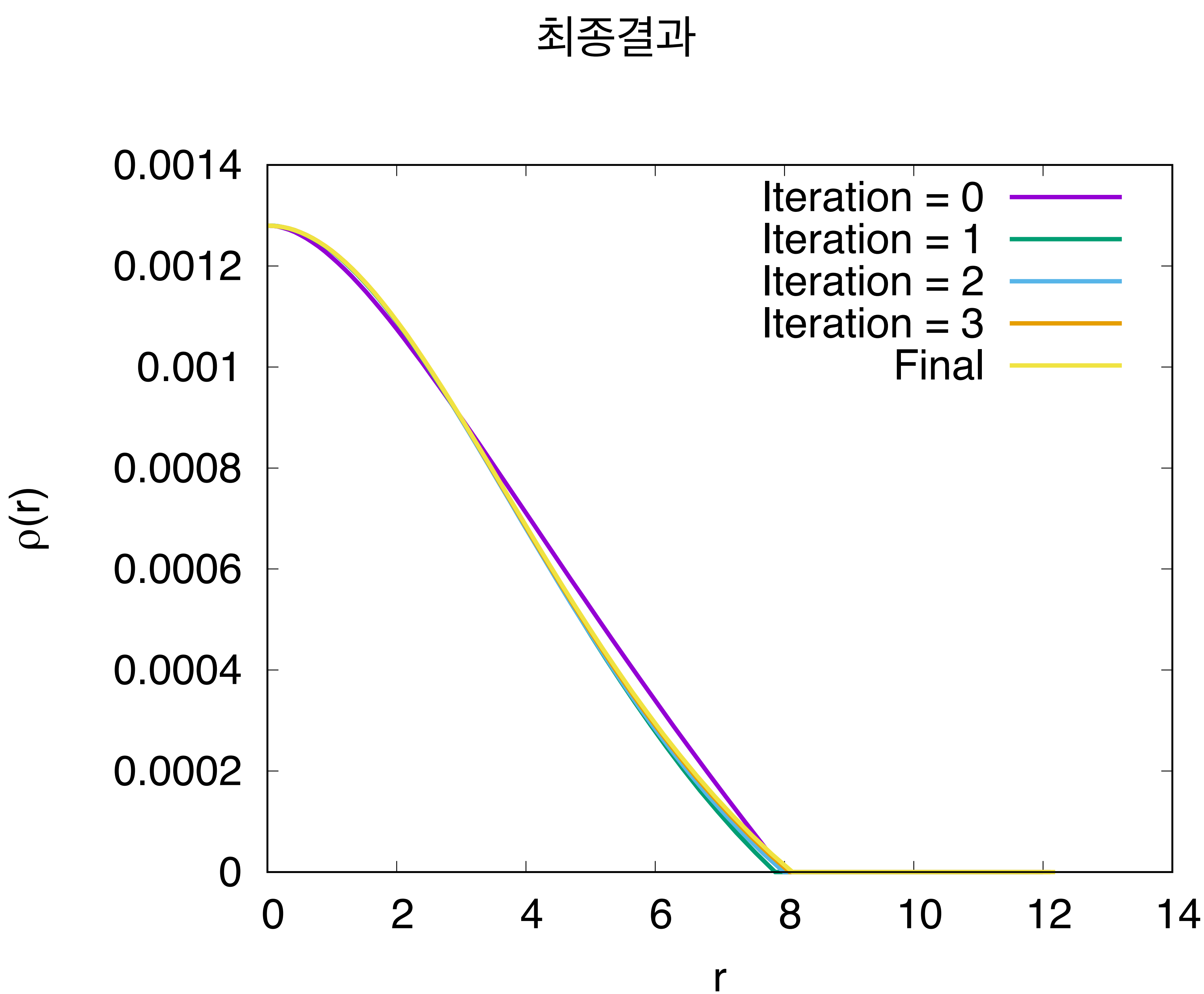
Solution procedure



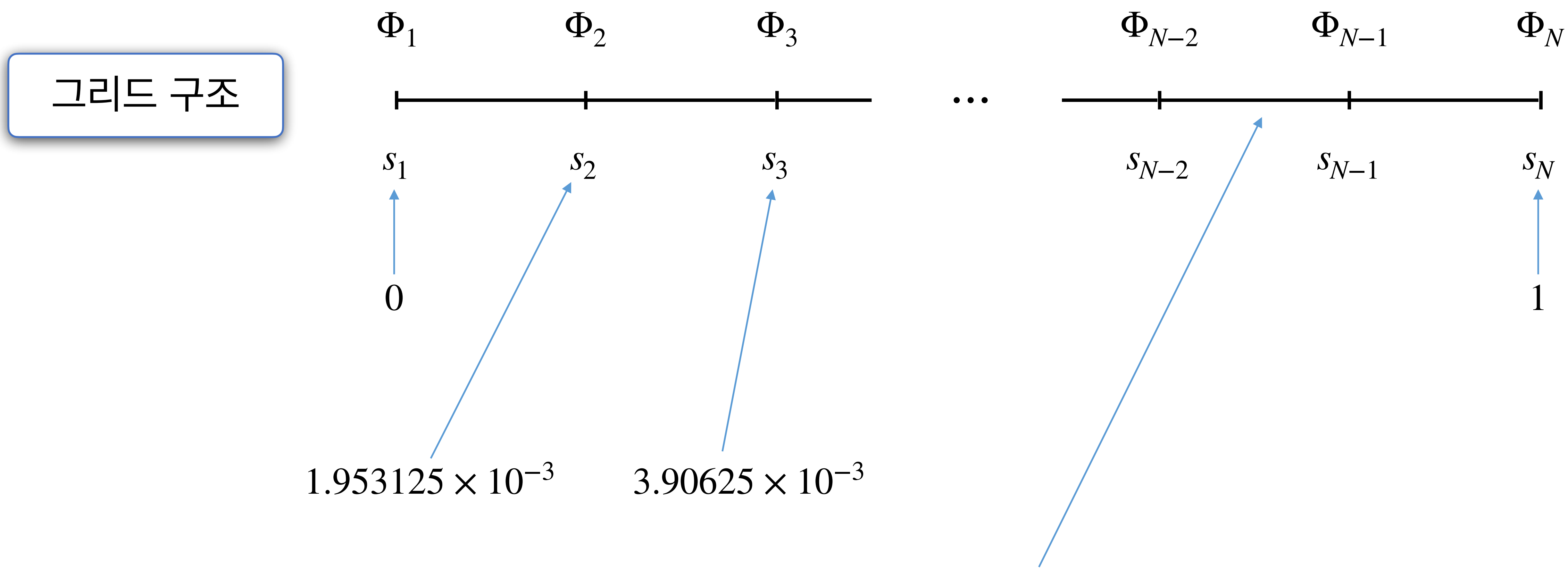
n	r_s^2	C	$\left\lvert \dfrac{[r_s^2]_n - [r_s^2]_{n-1}}{[r_s^2]_n} \right\rvert$	$\left\lvert \dfrac{C_n - C_{n-1}}{C_n} \right\rvert$
1	6.1716124772187584E+01	-1.8660983143549453E-01	3.7006134721563826E-02	1.0000000000000000E+00
2	6.3857598366076225E+01	-1.7632476946919279E-01	3.3535141450391277E-02	5.8330216436770881E-02
3	6.5261852274811176E+01	-1.7372033138927334E-01	2.1517224224984872E-02	1.4992131658345790E-02
4	6.5784162318657792E+01	-1.7301276928606002E-01	7.9397536646670635E-03	4.0896524928945308E-03
5	6.5949732907096632E+01	-1.7281959351806733E-01	2.5105573766625996E-03	1.1177885797566793E-03

⋮

Relative change $< \text{tol} = 1 \times 10^{-6}$
이 될 때까지 계산을 지속


최종결과
0.0014
0.0012
0.001
0.0008
0.0006
0.0004
0.0002
0
0
2
4
6
8
10
12
14
r
ρ(r)
Iteration = 0
Iteration = 1
Iteration = 2
Iteration = 3
Final

완화법 힌트



$$\Delta s = \frac{1}{N-1} = 1.953125 \times 10^{-3}, \quad N = 513$$

초기조건 $\Phi_i^1 = 0 \ \ (i = 1, \cdots N)$

↓

경계조건 $r_1^1 = -\Phi_3^1 + 4\Phi_2^1 - 3\Phi_1^1, \quad \Phi_N = 0$

↓

Residual $r_i^1 = \Phi_{i+1}^1 - 2\Phi_i^1 + \Phi_{i-1}^1 + \frac{1}{s_i}\left(\Phi_{i+1}^1 - \Phi_{i-1}^1\right)\Delta s - 4\pi r_s^2 \rho_i (1 - s_i)^{-4} \Delta s^2 \quad (i = 2, \cdots, N-1)$

↓

Update $\Phi_i^1 \rightarrow \Phi_i^2$ $\Phi_i^2 = \Phi_i^1 - r_i^1 / \left(\frac{\partial r_i^1}{\partial \Phi_i^1} \right)$

↓

초기조건 $\Phi_i^2 \ \ (i = 1, \cdots N)$

↓

경계조건 $r_1^2 = -\Phi_3^2 + 4\Phi_2^2 - 3\Phi_1^2, \quad \Phi_N = 0$

↓

Residual $r_i^2 = \Phi_{i+1}^2 - 2\Phi_i^2 + \Phi_{i-1}^2 + \frac{1}{s_i}\left(\Phi_{i+1}^2 - \Phi_{i-1}^2\right)\Delta s - 4\pi r_s^2 \rho_i (1 - s_i)^{-4} \Delta s^2 \quad (i = 2, \cdots, N-1)$

⋮

주의점